SPECTRAL THEORY

NELSON DUNFORD

Spectral theory has for its origin the classical theory of finite matrices and, in a sense, includes all theories which may be specialized to some particular phase of that classical theory. Such a meaning for the term spectral theory is perhaps too broad, for its acceptance would force one, as is becoming increasingly evident, to admit that spectral theory embraces not a small part of such fields of mathematics as the ergodic theory, the theory of probability, absolutely convergent trigonometric series, Fourier transforms, the theory of continuous functions on a bicompact Hausdorff space, the theory of partially ordered rings and Abelian groups, and perhaps even the well known embedding theorem concerning completely regular topological spaces. In fact, even though by far the greatest part of spectral theory that has been at present developed has for its roots the properties of a very special type of matrix, namely the Hermitian matrix, it is certainly true that modern developments have made significant contacts with all the theories just mentioned.

Although I shall touch briefly upon certain features of these modern developments my chief concern here will not be an attempt to describe the wealth of ideas that have grown out of the properties of an Hermitian matrix, but rather to show how the properties of a general matrix can guide the way to the solution of a special class of problems of vital interest in analysis. The type of problem with which I shall be concerned is illustrated by the following question. How can it be determined whether or not a given sequence of polynomials $f_n(T)$ in a linear operator T on a Banach space converges to a specified type of limit operator? Before making this problem more precise I should like to show how the elementary properties of a finite matrix can, if put in the proper form, give an immediate answer to all such questions of convergence for a finite dimensional space.

Suppose $\mathfrak{X}$ is an m-dimensional linear vector space over the field of complex numbers (or over any algebraically closed field). Let T be a linear operator in $\mathfrak{X}$, that is, T is an $m \times m$ matrix whose elements are complex numbers. For every complex number λ and every

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The address by Taylor which follows this was delivered on the same day before the Stanford meeting. To some extent the two papers overlap each other in subject matter, but they also complement each other through the divergence of the authors' points of view.

$n=0, 1, 2, \cdots$ define the linear manifolds

$$\mathfrak{N}_\lambda^n = (\lambda I - T)^n \mathfrak{X}, \qquad \mathfrak{M}_\lambda^n = \mathop{\mathcal{E}}_x [x \in \mathfrak{X}, (\lambda I - T)^n x = 0].$$

It is clear that for a fixed λ the manifold $\mathfrak{M}_\lambda^n$ is non-decreasing in n and the manifold $\mathfrak{N}_\lambda^n$ is nonincreasing. It is also clear that each λ defines uniquely a positive integer or zero, $\nu=\nu(\lambda)$, called the *index* of λ, such that $\mathfrak{M}_\lambda^\nu=\mathfrak{M}_\lambda^n$ for $n \geqq \nu$ while for $n<\nu$ we have $\mathfrak{M}_\lambda^n$ a proper subset of $\mathfrak{M}_\lambda^{n+1}$. It is also readily shown that $\mathfrak{N}_\lambda^\nu=\mathfrak{N}_\lambda^n$ for $n \geqq \nu$ while for $n<\nu$ we have $\mathfrak{N}_\lambda^{n+1}$ a proper subset of $\mathfrak{N}_\lambda^n$. If $n \geqq \nu(\lambda)$ we always have

$$\mathfrak{X} = \mathfrak{N}_\lambda^n \oplus \mathfrak{M}_\lambda^n, \tag{1}$$

and conversely if (1) holds then $n \geqq \nu(\lambda)$. Let $\lambda_1, \cdots, \lambda_k$ be the distinct characteristic numbers of T and let $\nu_1, \cdots, \nu_k$ be the corresponding indices, then

$$\mathfrak{X} = \mathfrak{M}_{\lambda_1}^{\nu_1} \oplus \cdots \oplus \mathfrak{M}_{\lambda_k}^{\nu_k}. \tag{2}$$

Equation (2) shows that there are uniquely defined projection operators (that is, idempotent matrices) $E_{\lambda_1}, \cdots, E_{\lambda_k}$ such that

$$I = E_{\lambda_1} + \cdots + E_{\lambda_k}, \quad E_{\lambda_i}\mathfrak{X} = \mathfrak{M}_{\lambda_i}^{\nu_i}, \quad E_{\lambda_i} \cdot E_{\lambda_j} = 0, \qquad i \neq j. \tag{3}$$

Since, in the space $\mathfrak{M}_{\lambda_i}^{\nu_i}$, any polynomial $f(T)$ in T is equal to its residue modulo $(T-\lambda_i I)^{\nu_i}$ we have from (3)

$$f(T) = \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{(T - \lambda_i I)^j}{j!} f^{(j)}(\lambda_i) E_{\lambda_i}. \tag{A}$$

While our chief purpose will be to investigate some of the consequences of this formula (A), or rather the formula corresponding to (A) in the infinite case, we should like to point out very briefly the relation of this formula to the special case which has led to so many fruitful investigations. If the indices ν_i of T are all 1, in particular if T is an Hermitian matrix, then (A) becomes

$$f(T) = \sum_{i=1}^{k} f(\lambda_i) E_{\lambda_i}. \tag{B}$$

The important algebraic difference between cases (A) and (B) is that, as is easily seen, in case (B) the ring of polynomials in T is isomorphic with a direct sum of fields, that is, isomorphic with a linear space of scalar functions, while in case (A) the ring of all polynomials in T is isomorphic with a direct sum of reduced polynomial rings. Another

way of seeing the chief difference between the two cases is to observe that in case (B) we have no nilpotent elements, that is, no radical in the ring, while in case (A) we may have $f^m(T)=0$, $f(T)\neq 0$. It is formula (B) which symbolizes the classical theory of reduction of quadratic forms. This theory was extended to infinite forms by Hilbert and his school and by E. H. Moore and his students. It is perhaps just to say that it is F. Riesz to whom we owe the advantages of the concept of a linear operator. To F. Riesz we owe the formula

$$(B') \qquad f(T) = \int_{-\infty}^{+\infty} f(\lambda) dE_\lambda$$

which replaces (B) in the case of a bounded self-adjoint operator in Hilbert space. It was J. von Neuman and M. H. Stone who developed the theory symbolized by (B′) in the case where T is an unbounded self-adjoint operator in Hilbert space. In recent years the formula (B′) has been rapidly disappearing from the theory of quadratic forms. In fact, the theory of bounded quadratic forms has become, in the hands of Stone,[1] Gelfand,[2] Kakutani,[3] and others, almost identical with the theory of continuous real functions on a bicompact Hausdorff space. Each of these mathematicians has, I believe, taken roughly the point of view just mentioned, that is, that a commutative family of Hermitian matrices may be thought of as a family of scalar functions and each one has been led to a characterization of the family of *all* real continuous functions on a bicompact Hausdorff space. While their starting points are somewhat different there are, in each case, postulates sufficient to rule out a radical in the system, that is, these mathematicians have all been working with systems generalizing formula (B).

With this very incomplete summary of what has been done with formula (B) let us return to (A) and ask the question:—when does a given sequence $f_n(T)$ of polynomials in a general matrix T converge? We shall describe two ways of answering this question, the first being fairly obvious, the second less so but in many ways more practical than the first. The first method is illustrated by the following theorem.

THEOREM 1. *The sequence $f_n(T)$ converges if and only if $f_n^{(\rho)}(\lambda_i)$ con-*

[1] M. H. Stone, *A general theory of spectra* I, Proc. Nat. Acad. Sci. U. S. A. vol. 26 (1940) pp. 280–283.

[2] I. Gelfand, *Normierte Ringe*, Recueil Math. (Mat. Sbornik) N.S. vol. 51 (1941) pp. 3–24.

[3] S. Kakutani, *Concrete representation of abstract (M) spaces*, Ann. of Math. vol. 42 (1942) pp. 994–1024.

verges for each $i=1, \cdots, k$ *and* $j=0, \cdots, \nu_i-1$. *Also* $f_n(T)\to f(T)$ *if and only if* $\lim_n f_n^{(j)}(\lambda_i)=f^{(j)}(\lambda_i)$, $i=1, \cdots, k$, $j=0, \cdots, \nu_i-1$.

The sufficiency of the conditions is obvious from (A) while their necessity is an immediate consequence of the definition of ν_i.

We shall reserve any general statement of the second method until later and here merely illustrate it by a typical example.

EXAMPLE. *The sequence* $(\sum_{r=0}^{n-1}T^r)/n$ *converges if and only if* $T^n/n\to 0$.

The truth of this statement is readily seen by applying Theorem 1. For if $f_n(\lambda)=\lambda^n/n$ then $f_n(\lambda)\to 0$ if and only if $|\lambda|\leqq 1$. Whereas all derivatives $f_n^{(j)}(\lambda)\to 0$ if $|\lambda|<1$, not even the first derivative $f^{(1)}(\lambda)\to 0$ if $|\lambda|=1$. Thus by Theorem 1 we can say that $T^n/n\to 0$ if and only if all roots λ_i of T have $|\lambda_i|\leqq 1$, and also every root λ_i of T with $|\lambda_i|=1$ has the corresponding index $\nu_i=1$. A similar application of Theorem 1 shows that $(\sum_{r=0}^{n-1}T^r)/n$ converges if and only if the spectral points of T satisfy precisely the same conditions.

It is not pure coincidence that these two statements about convergence have precisely the same spectral interpretation. It is a consequence of a general principle which will be discussed more fully later.

The formal difference that we wish to emphasize between the theorem and the example is obvious; namely, in the example there is no mention of the spectrum of T. It is for this reason that a theorem of the type of the example, that is, a theorem asserting the equivalence of two statements about convergence, is in many ways more practical than one like Theorem 1. For example if a deck of cards is repeatedly shuffled in such a way that in any given shuffle there is a definite probability p_{ij} that the card in the ith place goes into the card in the jth place, and this probability is fixed and independent of preceding shuffles, then one knows immediately that the probability of finding in the jth place, after n shuffles, the card which was originally in the ith place is the element $p_{ij}^{(n)}$ of the nth iterate of the matrix $P=(p_{ij})$. Hence it is seen without any spectral considerations that $P^n/n\to 0$. Similarly there are many examples of linear operators T in analysis where for one reason or another (perhaps because $|T^n|\leqq M$) it is known that $T^n/n\to 0$, and hence a theorem of the type of the example could be much more easily applied than one like Theorem 1.

Our main objective will now be the description of a method for arriving at a class of theorems similar to the example, that is, theorems asserting the equivalence of two statements about convergence,

but in the case where T is a bounded linear operator on a complete normed linear vector space not necessarily of finite dimension. In this general situation there are several different but closely related problems, for there are usually three and sometimes four different meanings for convergence. We always have the notions of convergence in the uniform, strong, and weak topologies, and, in case the space in question consists of measurable functions we have the notion of convergence almost everywhere. I wish to outline a unified theory of convergence, in each of the four types, of a sequence $f_n(T)$ of functions of an operator T to a *specified* type of limit operator.

In brief our method for arriving at this goal consists of two steps. The first and easiest step is the establishment of an operational calculus to replace formula (A). The second and more difficult one is the matter of applying the operational calculus. Here the chief tools are theorems corresponding to (a) the minimal equation theorem for matrices, (b) the theorem of Sylvester concerning the determinant of a polynomial in a matrix and (c) the various decompositions of the whole space into a direct sum of its subspaces determined by the spectrum of a matrix.

Returning our attention for a moment to a finite matrix T and in particular to equation (A) we observe that one formula, and perhaps the only one, which may be used to replace (A) and which does not in any way exhibit the finite character of the space $\mathfrak{X}$ or the operator T is

$$(A') \qquad f(T) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)d\lambda}{\lambda I - T},$$

where C is a set of small circles $C_1, \cdots, C_k$ with C_i surrounding λ_i. That formula (A′) is valid for any polynomial $f(T)$ may be verified by direct computation or it may be derived from (A) by using the Hermite interpolation formula. Conversely, it is not difficult to pass from (A′) to (A) without going through the preliminary algebraic arguments used to derive (A). The reason for taking C as a set of small circles each one of which surrounds one and only one spectral point of T rather than taking C as one large circle containing the whole spectrum is the following. It is essential for our purposes that we do not have to restrict the operational calculus expressed by (A′) to the class of functions $f(\lambda)$ which are regular and single-valued on a *connected domain* containing the spectrum of T. We shall consider the class $\mathcal{F}(T)$ of functions $f(\lambda)$ which are regular and single-valued in the closure of a domain $\mathcal{D}$ (which may vary with the function $f \in \mathcal{F}(T)$) satisfying the conditions

(i) $\mathcal{D}$ is a finite sum of connected open sets $\mathcal{D}_i$ with $\overline{\mathcal{D}}_i\overline{\mathcal{D}}_j=0$, $i\neq j$.

(ii) The boundary C of $\mathcal{D}$ consists of a finite number of disjoint closed rectifiable Jordan curves contained in the resolvent set $\rho(T)$ of T.

(iii) The spectrum $\sigma(T)$ of T is contained in $\mathcal{D}$.

A domain $\mathcal{D}$ satisfying (i) and (ii) but not necessarily (iii) will be called a *T-admissible domain*, in symbols $\mathcal{D}=\mathcal{D}(T)$. These are the requirements whether T be a finite matrix, a bounded linear operator on a complex Banach space, or an element of a normed ring. Thus for example in the case of the finite matrix we may take $f(\lambda)\equiv 1$ in and on C_i and $f(\lambda)\equiv 0$ in and on C_j, $j\neq i$, and if we do the result is

$$E_{\lambda_i}=\frac{1}{2\pi i}\int_{C_i}\frac{d\lambda}{\lambda I-T}.$$

This result, as well as the general one expressed by (A′), was discovered in 1928 by Fantappié,[4] who gave (A′) as a basis for an operational calculus on finite matrices. Fantappié proved the standard rules for such a calculus, namely,

(α) $(\alpha f+\beta g)(T)=\alpha f(T)+\beta g(T)$.

(β) $(f\cdot g)(T)=f(T)\cdot g(T)$.

(γ) If $f(\lambda)=\sum_{n=1}^{m}\alpha_n\lambda^n$ then $f(T)=\sum_{n=1}^{m}\alpha_n T^n$.

It was pointed out by E. Cartan[5] that formula (A′) could be used as a basis for an operational calculus on infinite matrices. In what follows we shall think of T as a bounded linear operator on a complex Banach space and shall write $\rho(T)$ for the resolvent set of T, that is, the set of complex numbers for which $R_\lambda(T)=(\lambda I-T)^{-1}$ exists as an everywhere defined and hence bounded linear operator. The symbol $\sigma(T)$ will be used for the spectrum of T, that is, the complex numbers not in $\rho(T)$. It should be recalled that the index $\nu=\nu(\lambda)$ (as defined for finite matrices) of a point $\lambda\in\sigma(T)$ may be 0 or ∞. Recalling that $R_\lambda(T)$ is analytic on $\rho(T)$ we define for every $f\in\mathcal{F}(T)$ the operator $f(T)$ by the formula (A′), where the integral is taken around the boundary C of any T-admissible domain $\mathcal{D}$ which contains $\sigma(T)$ and

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[4] L. Fantappié, *La calcul des matrices*, C. R. Acad. Sci. Paris vol. 186 (1928) pp. 619–621. In the case where $f(\lambda)$ is a power series and C is a circle, formula (A′) has been given by H. Poincaré, *Sur les groupes continus*, Transactions of the Cambridge Philosophical Society vol. 18 (1900) pp. 220–255.

[5] The suggestion was made in a letter to G. Giorgi. For this reference as well as for others pertaining to an operational calculus see C. C. MacDuffee, *The theory of matrices*, Berlin, 1933.

upon which $f(\lambda)$ is regular and single-valued. We shall add to the basic rules (α), (β), (γ) of Fantappié two more, namely,

(δ) $f(\sigma(T))=\sigma(f(T)), f\in\mathcal{F}(T)$.

(ϵ) If $f\in\mathcal{F}(T)$, $g\in\mathcal{F}(f(T))$, and $F(\xi)=g(f(\xi))$ then $F\in\mathcal{F}(T)$ and $F(T)=g(f(T))$.

A frequently used corollary of (β) is the following. If $f\in\mathcal{F}(T)$ is identically 1 in a neighborhood of a spectral set σ of T (a spectral set σ of T is a subset of $\sigma(T)$ which is open and closed in $\sigma(T)$) and identically zero in a neighborhood of the complement $\sigma'=\sigma(T)-\sigma$ of σ then $f(T)$ is a projection. In other words the operator $E_\sigma[T]=(1/2\pi i)\int_C R_\lambda(T)d\lambda$, where C is the boundary of a T-admissible domain $\mathcal{D}$ with $\sigma=\mathcal{D}\cdot\sigma(T)$, is a projection. This result was proved recently by Lorch[6] who used methods independent of an operational calculus. Lorch also showed that the Boolean algebra of all such projections is isomorphic with the Boolean algebra of all spectral sets of T. The fact that the mapping $\sigma\to E_\sigma[T]$ is a homomorphism is readily seen from (α), (β). That it is actually an isomorphism follows from the following fundamental principle.

THEOREM 2. *When T is considered as an operator in the space $\mathfrak{X}_\sigma=E_\sigma[T]\mathfrak{X}$ it has σ for its spectrum. Furthermore for $\lambda\in\sigma$ any one of the following statements is true if and only if it is true when T is regarded as an operator in $\mathfrak{X}_\sigma$: λ is in the point spectrum of T, λ is in the residual spectrum of T, λ is in the continuous spectrum of T, λ is a pole of order ν for $R_\xi(T)$, λ is an essential singularity for $R_\xi(T)$.*

This is all we shall say about the formal operational calculus. The finite dimensional space has been replaced by a complex Banach space $\mathfrak{X}$, the matrix by a linear operator T in $\mathfrak{X}$, the class of polynomials by the class $\mathcal{F}(T)$, and the formula (A) by the formula (A′). Now it is perfectly evident that the arguments used to derive Theorem 1 from formula (A) can not be applied immediately to formula (A′). Certain elementary facts can, however, be stated immediately. On the one hand, as a necessary condition for the convergence of a sequence $f_n(T)$ where $f_n\in\mathcal{F}(T)$, we have from (α) and (δ) that $|f_n(\lambda)-f_m(\lambda)|\leqq|f_n(T)-f_m(T)|$, $\lambda\in\sigma(T)$, so that if $f_n(T)$ converges in the uniform topology of operators to an operator U we must have $f_n(\lambda)$ converging uniformly for $\lambda\in\sigma(T)$ to a continuous function $f(\lambda)$ defined for $\lambda\in\sigma(T)$. This function $f(\lambda)$ which will be mentioned later is called the *spectral function* of U and depends only on U and not on the par-

[6] E. R. Lorch, *The spectrum of linear transformations*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 238–248.

ticular sequence $f_n(T)$ with $f_n \in \mathcal{F}(T)$ used to approach U. On the other hand it is clear from (A′) that the uniform convergence of $f_n(\lambda)$ on a *fixed* (that is, independent of n) T-admissible domain containing $\sigma(T)$ is a sufficient condition for the convergence of $f_n(T)$. Unfortunately, however, most of the interesting problems in analysis of this type are concerned with a sequence $f_n(\lambda)$ whose points of divergence have a spectral point of T as a limiting point. Take for example the ergodic theorem and suppose that $\lambda = 1$ is in the spectrum $\sigma(T)$ (this is the only non-trivial case). The sequence of polynomials $(\sum_{\nu=0}^{n-1} \lambda^\nu)/n$ clearly diverges at points in every neighborhood of $\lambda = 1$. Thus it seems necessary to seek for more refined methods. To this end we shall virtually eliminate formula (A′) by deriving from it certain fundamental results of an algebraic nature which can be used as a basis for the convergence theory. We shall describe here three of the principal theorems of this type.

The minimal equation theorem. For a finite matrix T this theorem is well known and asserts that a polynomial $f(T)$ in T vanishes if and only if $f(\lambda)$ contains $\prod_{i=1}^{k} (\lambda - \lambda_i)^{\nu_i}$ as a factor. The statement for a general linear operator is almost identical and reads as follows.

THEOREM 3 (THE MINIMAL EQUATION THEOREM). *If $f \in \mathcal{F}(T)$ then $f(T) = 0$ if and only if*

(i) *For every pole λ of $R_\xi(T)$ of order ν, $f^{(j)}(\lambda) = 0, j = 0, 1, \cdots, \nu - 1$.*

(ii) *$f(\lambda) \equiv 0$ on a neighborhood containing all the spectrum $\sigma(T)$ except perhaps poles of $R_\xi(T)$.*

A corollary of this is

COROLLARY 1. *If $f, g \in \mathcal{F}(T)$ then $f(T) = g(T)$ if and only if*

(i) *For every pole λ of $R_\xi(T)$ of order ν, $f^{(j)}(\lambda) = g^{(j)}(\lambda), j = 0, \cdots, \nu - 1$.*

(ii) *$f(\lambda) = g(\lambda)$ for every λ in a neighborhood containing all the spectrum $\sigma(T)$ except perhaps poles of $R_\xi(T)$.*

COROLLARY 2. *Let $\lambda_1, \cdots, \lambda_k$ be poles of $R_\xi(T)$ of orders $\nu_1, \cdots, \nu_k$ respectively. Let σ' be the complement of the spectral set $\sigma = (\lambda_1, \cdots, \lambda_k)$. Then for every $f \in \mathcal{F}(T)$ we have*

$$f(T) = f(T)E_{\sigma'}[T] + \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{(T - \lambda_i I)^j}{j!} f^{(j)}(\lambda_i) E_{\lambda_i}.$$

Corollary 2 follows from Corollary 1 as follows. If $\psi(\lambda)$ is the characteristic function of a small neighborhood of λ_i then by Corollary 1 and (β) we have $\psi(T)f(T) = \psi(T)P(T) = E_{\lambda_i}[T]P(T)$ where $P(\lambda) = \sum_{j=0}^{\nu_i - 1} (\lambda - \lambda_i)^j f^{(j)}(\lambda_i)/j!$

As is suggested by these theorems we have also the fact that if λ is a pole of order ν for $R_\xi(T)$ then λ has index ν and also the manifold $\mathfrak{N}_\lambda^\nu$ is closed and $\mathfrak{N}_\lambda^\nu = \mathfrak{N}_\lambda^n$ for $n \geqq \nu$, while $\mathfrak{N}_\lambda^{n+1}$ is a proper subset of $\mathfrak{N}_\lambda^n$ for $n < \nu$. Just as in the case of finite matrices (see equation (1)) there is the decomposition[7]

$$\mathfrak{X} = \mathfrak{N}_\lambda^n \oplus \mathfrak{M}_\lambda^n, \qquad n \geqq \nu, \tag{1'}$$

and *conversely if for some n we have* $\mathfrak{N}_\lambda^n$ *a closed complement of* $\mathfrak{M}_\lambda^n$ *then* λ *is a pole of order* $\nu \leqq n$.

When λ is a pole of order ν we always have $\mathfrak{M}_\lambda^\nu = \mathfrak{X}_\lambda \equiv E_\lambda[T]\mathfrak{X}$. When λ is not a pole but has index ν we frequently have the decomposition

$$\mathfrak{X} = \overline{\mathfrak{N}_\lambda^\nu} \oplus \mathfrak{M}_\lambda^\nu. \tag{1''}$$

The Sylvester theorem. This theorem generalizes the formula $\sigma(f(T)) = f(\sigma(T))$. For finite matrices this formula merely means that if $\lambda_1, \cdots, \lambda_k$ are the distinct roots of an $n \times n$ matrix T then $f(\lambda_1), \cdots, f(\lambda_k)$ are the roots of $f(T)$. A theorem due to Sylvester asserts more, namely, if $\lambda_1, \cdots, \lambda_k$ are the distinct roots of T and if m_i is the multiplicity of λ_i then

$$\text{Det } (f(T) - \lambda I) = \prod_{i=1}^{k} (f(\lambda_i) - \lambda)^{m_i}. \tag{i}$$

This means that if $\lambda_1, \cdots, \lambda_n$ are the roots of T, each repeated according to its multiplicity, then $f(\lambda_1), \cdots, f(\lambda_n)$ are the roots of $f(T)$, where the number of repetitions of a given number in this array is its multiplicity as a root of $f(T)$. Recalling that the multiplicity $m(\lambda, T)$ of a root λ of T of index ν is the number of linearly independent solutions of $(\lambda I - T)^\nu x = 0$, that is, $m(\lambda, T)$ is the dimension of $\mathfrak{M}_\lambda^\nu = \mathfrak{X}_\lambda[T] = E_\lambda[T]\mathfrak{X}$, we see that the Sylvester Theorem, that is, the formula

$$m(\mu, f(T)) = \sum_{\lambda \in \sigma} m(\lambda, T), \qquad \sigma = f^{-1}(\mu), \tag{ii}$$

is an immediate consequence of

$$\mathfrak{X}_\mu[f(T)] = \sum_{\lambda \in \sigma} \mathfrak{X}_\lambda[T], \qquad \sigma = f^{-1}(\mu), \tag{iii}$$

(where the sum on the right is a vector sum). Whereas formula (iii)

[7] At this point reference should be made to F. Riesz, *Über lineare Funktionalgleichungen*, Acta Math. vol. 41 (1918) pp. 71–98, who proved similar theorems for a compact operator.

states the equality of two manifolds, formula (ii) merely states the equality of their dimensions. A more desirable form of this theorem than (iii) is

$$\text{(iv)} \qquad E_\mu[f(T)] = \sum_{\lambda \in \sigma} E_\lambda[T] = E_\sigma[T],$$

which states the equality of two projections rather than, as (iii) does, the equality of the manifolds upon which they project. It is, roughly speaking, the form (iv) that the Sylvester theorem takes in the infinite case. To be more precise we have

THEOREM 4 (THE SYLVESTER THEOREM). *Let $f \in \mathcal{F}(T)$ and let τ be a spectral set of $f(T)$. Then $\sigma = \sigma(T) \cdot f^{-1}(\tau)$ is a spectral set of T and $E_\tau[f(T)] = E_\sigma[T]$.*

This theorem has a further generalization which will be needed later. Let $\mathcal{R}(T)$ be the ring of all operators $f(T)$ where $f \in \mathcal{F}(T)$ and let $\overline{\mathcal{R}(T)}$ be the ring of all operators which are limits in the uniform topology of operators of elements in $\mathcal{R}(T)$. Then, as we have seen earlier, with each $U \in \overline{\mathcal{R}(T)}$ is associated a unique continuous function $f(\lambda)$ defined on $\sigma(T)$. This spectral function also has the property that $f(\sigma(T)) \subset \sigma(U)$. Of course if $U = g(T) \in \mathcal{R}(T)$ then $f(\lambda) = g(\lambda)$ on $\sigma(T)$. In terms of these notions we have

THEOREM 4′ (THE GENERAL SYLVESTER THEOREM). *Let f be the spectral function of an operator $U \in \overline{\mathcal{R}(T)}$ and let τ be a spectral set of U. Then $\sigma = f^{-1}(\tau)$ is a spectral set of T and $E_\sigma[T] = E_\tau[U]$.*

A typical decomposition theorem. There are many types of decomposition theorems that seem to be needed for a general theory of convergence. We shall content ourselves here with one illustration. This theorem to follow generalizes the theorem mentioned for finite matrices in connection with equation (1), and its comprehension needs the following concepts. Let $f \in \mathcal{F}(T)$ and define the manifolds

$$\mathfrak{N}[f] = f(T)\mathfrak{X}, \qquad \mathfrak{M}[f] = \mathop{\mathcal{E}}_{x} [x \in \mathfrak{X}, f(T)x = 0].$$

THEOREM 5. *Let $f \in \mathcal{F}(T)$ and suppose that $f(\lambda)$ is not identically zero on any of the domains in which it is regular. Let $\lambda_1, \cdots, \lambda_k$ be the roots of $f(\lambda)$ and $m_1, \cdots, m_k$ their multiplicities. Then the following statements are equivalent.*

(1) *$\mathfrak{X} = \mathfrak{M}[f] \oplus \mathfrak{N}[f]$, $\mathfrak{N}[f]$ is closed.*

(2) *For $i = 1, \cdots, k$, λ_i is either in the resolvent set $\rho(T)$ or else a pole of order $\nu_i \leqq m_i$ of $R_\xi(T)$.*

(3) *The finite set* $\sigma=(\lambda_1, \cdots, \lambda_k)\sigma(T)$ *is a spectral set of* T *and* $\mathfrak{X}_\sigma=\mathfrak{M}[f]$, $\mathfrak{X}_{\sigma'}=\mathfrak{N}[f]$.

When the roots λ_i of $f(\lambda)$ are not poles of $R_\xi(T)$, perhaps not even isolated in the spectrum $\sigma(T)$, but have indices $\nu_i \leqq m_i$ we often have a decomposition of the form $\mathfrak{X}=\mathfrak{M}[f]\oplus\overline{\mathfrak{N}[f]}$.

Some convergence theorems. While it is clear that a certain type of convergence theorem may be obtained from Corollary 2 of the minimal equation theorem just as Theorem 1 was obtained from formula (A), we wish to restrict our attention here to a special case which presents unexpected features not found in the general situation. We shall be interested in when a given sequence $f_n(T)$, $f\in F(T)$, that is, a given sequence $f_n(T)\in R(T)$, converges to a special type of projection E. We shall first discuss the case of convergence in the uniform topology of operators. As simple examples show we do not in general have $\mathfrak{R}(T)=\overline{\mathfrak{R}(T)}$, that is, in general $U=\lim_n f_n(T)$ is not in $\mathfrak{R}(T)$. However in case U is a projection E it is necessarily in $\mathfrak{R}(T)$. This is the first of the two fundamental necessity theorems, and it is the necessity of the conditions in the case of convergence in the uniform topology of operators that presents the most difficulty.

THEOREM 6 (THE FIRST NECESSITY THEOREM). *Every projection* $E\in\overline{\mathfrak{R}(T)}$ *is in* $\mathfrak{R}(T)$ *and* $E=E_\sigma[T]$ *where* σ *is a spectral set of* T *and consists of all* $\lambda\in\sigma(T)$ *where the spectral function* $f(\lambda)$ *of* E *has the value* 1. *Thus* $E=0$ *if and only if there are no such* λ.

This theorem is an immediate consequence of the general Sylvester theorem (Theorem 4′). To see this we observe the readily verified fact that the resolvent $R_\xi(E)$ of a projection E is given by the equation $R_\xi(E)=E/(\xi-1)+(I-E)/\xi$, $\xi\neq 0, 1$. Thus $E_1[E]=E$, and if we take τ in Theorem 4′ to be the point $\lambda=1$ then $E=E_1[E]=E_\sigma[T]$ where σ consists of all $\lambda\in\sigma(T)$ with $f(\lambda)=1$.

THEOREM 7 (THE SECOND NECESSITY THEOREM). *Let* P *be a polynomial not identically zero and let* $f_n\in\mathcal{J}(T)$. *If* $f_n(T)$ *converges to a projection* E *with* $E\mathfrak{X}\subset\mathfrak{M}[P]$ *then either there are no* $\lambda\in\sigma(T)$ *where* $f_n(\lambda)\to 1$ *in which case* $E=0$ *or else*

(1) *The set* σ *of points* $\lambda\in\sigma(T)$ *where* $f_n(\lambda)\to 1$ *consists of a finite number of poles* $\lambda_1, \cdots, \lambda_k$ *of* $R_\xi(T)$.

(2) *If* $\nu_1, \cdots, \nu_k$ *are the orders of the poles* $\lambda_1, \cdots, \lambda_k$ *then* $f_n(\lambda_i)\to 1$, $f_n^{(j)}(\lambda_i)\to 0$, $j=1, \cdots, \nu_i-1$, $i=1, \cdots, k$.

(3) *For* $i=1, \cdots, k$, λ_i *is a root of* $P(\lambda)$ *and its multiplicity is at least* ν_i.

(4) $\prod_{i=1}^{k}(\lambda_i I-T)^{\nu_i}E=0$.

This theorem is a consequence of the minimal equation theorem together with the preceding theorem. The proof goes as follows. Let σ be the set of $\lambda\in\sigma(T)$ where $f_n(\lambda)\to 1$. Then, by Theorem 7, σ is a spectral set of T and $E=E_\sigma[T]$. Since $E\mathfrak{X}\subset\mathfrak{M}[P]$ we have $P(T)E_\sigma[T]=0$ and so (1), (3) and (4) follow immediately from the minimal equation theorem whereas (2) follows from its Corollary 2.

We have now outlined all of the difficult steps encountered in the proof of the following basic theorem. We shall omit the rest of the details.[8]

THEOREM 8. *Let* $P(\lambda)=\prod_{i=1}^{k}(\lambda_i-\lambda)^{\nu_i}$ *be a polynomial whose distinct roots are* $\lambda_1, \cdots, \lambda_k$. *Let* $f_n\in\mathcal{F}(T)$ *satisfy*

(1) $f_n(\lambda_i)\to 1$, $f_n^{(j)}(\lambda_i)\to 0$, $i=1, \cdots, k$, $j=1, \cdots, \nu_i-1$.

(2) $P(T)f_n(T)\to 0$.

Then the following assertions are equivalent.

(3) $f_n(T)\to E$, $E^2=E$, $E\mathfrak{X}=\mathfrak{M}[P]$.

(4) *For each* $i=1, \cdots, k$, λ_i *is either in* $\rho(T)$ *or else a pole of* $R_\xi(T)$.

(5) *For* $i=1, \cdots, k$, λ_i *is either in* $\rho(T)$ *or else a pole of* $R_\xi(T)$ *of order at most* ν_i.

(6) $\mathfrak{X}=\mathfrak{N}[P]\oplus\mathfrak{M}[P]$, $\mathfrak{N}[P]$ *is closed.*

(7) $\mathfrak{N}_{\lambda_i}^{\nu_i+1}$, $i=1, \cdots, k$, *is closed.*

It is the two assertions "$P(T)f_n(T)\to 0$" and "$f_n(T)$ converges" (for if $f_n(T)$ converges it must, in view of (1), converge to a projection on $\mathfrak{M}(P)$) which in this situation (that is, in view of (1) and (4) for example) have precisely the same spectral significance. This theorem serves to illustrate the general principle mentioned in connection with the example following Theorem 1, for if $P(T)=I-T$ and $f_n(T)=(\sum_{\nu=0}^{n-1}T^\nu)/n$ then the condition $P(T)f_n(T)\to 0$ is precisely $T^n/n\to 0$. Thus for a transformation T which satisfies either

(4′) The point $\lambda=1$ is either in the resolvent set $\rho(T)$ or a pole of $R_\xi(T)$,

or

(7′) $(I-T)^2\mathfrak{X}$ is closed,

we can say that the uniform ergodic theorem holds if and only if $T^n/n\to 0$, and conversely if the uniform ergodic theorem holds then

[8] The details of many of the results discussed here appear in the Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217.

so do (4′) and (7′). In fact if the uniform theorem holds we have

(6′) $(I-T)\mathfrak{X}$ is closed,

and, as we shall see, this condition is also necessary and sufficient for the uniform theorem providing the strong (that is, mean) theorem holds and $T^n/n\to 0$.

We shall now pass to the case of strong convergence and state

THEOREM 9. *Let* $P(\lambda)=\prod_{i=1}^{k}(\lambda_i-\lambda)^{\nu_i}$ *be a polynomial whose distinct roots are* $\lambda_1, \cdots, \lambda_k$. *Let* $f_n\in\mathcal{J}(T)$ *satisfy*

(1) $f_n(\lambda_i)\to 1, f_n^{(j)}(\lambda_i)\to 0, i=1, \cdots, k, j=1, \cdots, \nu_i-1$.

(2) $P(T)f_n(T)\to 0$ *strongly.*

Then the following assertions are equivalent.

(3) $f_n(T)\to E$ *strongly,* $E^2=E$, $E\mathfrak{X}=\mathfrak{M}[P]$.

(4) $f_n(T)x$ *is weakly compact,* $x\in\mathfrak{X}$.

(5) $\mathfrak{X}=\mathfrak{N}[P]\oplus\mathfrak{M}[P]$, $|f_n(T)|\leqq M$.

Thus in a reflexive space if (1) and $|f_n(T)|\leqq M$ are satisfied we can say that (2) and (3) are equivalent. This is another example where two statements about convergence have the same spectral significance. Taking $P(T)=I-T, f_n(T)=(\sum_{\nu=0}^{n-1}T^\nu)/n$, as before, we can say that in a reflexive space if $|(\sum_{\nu=0}^{n-1}T^\nu)/n|\leqq M$ then $(\sum_{\nu=0}^{n-1}T^\nu)/n$ converges strongly if and only if $T^n/n\to 0$ strongly. The usual condition $|T^n|\leqq M$ assumed in ergodic theorems is not a necessary one. Hille and Szegö[9] have recently constructed operators satisfying the strong (that is, mean) as well as the almost everywhere ergodic theorem, and for which $|T^n|$ is of the order of $n^{1/4}$. The essential algebraic difference between the strong and the uniform theorem is seen by a comparison of (6) of Theorem 8 and (5) of Theorem 9.

We shall omit any discussion of weak convergence and pass to the case of almost everywhere convergence. Here we have as a basis a space S of points t. The space $\mathfrak{X}$ consists of scalar functions which are defined on S and which are measurable with respect to a completely additive non-negative measure function $|e|$ defined for e in a Borel field including S. It is assumed that $x=0$ if and only if $x(t)=0$ almost everywhere, and that addition and scalar multiplication of the functions correspond to the similar operations in $\mathfrak{X}$. It is assumed that if $x_n\to x$ in $\mathfrak{X}$ and $x_n(t)\to y(t)$ almost everywhere then $x(t)=y(t)$ almost everywhere. It is also assumed in case $|S|=\infty$ that S is the denumerable union of sets of finite measure. For a linear operator U in $\mathfrak{X}$ we shall write $U(x, t)$ for the value of the function Ux at the point t.

[9] E. Hille and G. Szegö, *Remarks on ergodic theorems*, to appear in the Trans. Amer. Math. Soc.

A fundamental theorem of Banach[10] enables us to state the following theorem as an immediate corollary of Theorem 9.

THEOREM 10. *Let P, f_n, E be as in the preceding theorem and satisfy* (1), (2), (4) *of that theorem. Then*

(1) *For every $x \in \mathfrak{X}$ we have* $\lim_n f_n(T)(x, t) = E(x, t)$ *almost everywhere if and only if*

(2) *For every $x \in \mathfrak{X}$* $\limsup_n |f_n(T)(x, t)| < \infty$ *almost everywhere and*

(3) *For every x in a fundamental set in $\mathfrak{X}$ the* $\lim_n f_n(T)P(T)(x, t)$ *exists almost everywhere.*

Again let us interpret this theorem by taking $P(T) = I - T$, $f_n(T) = (\sum_{\nu=0}^{n-1} T^\nu)/n$ where T is now a linear operator in the Lebesgue space $L(S)$, $|S| < \infty$, defined by a 1-1 point map ϕ of S into itself according to the formula $Tf = g$, $g(t) = f(\phi t)$. Clearly (3) is trivially satisfied with no restrictions on ϕ by merely taking the fundamental set to be the set of bounded measurable functions. Thus (3) is redundant and (2) becomes the necessary and sufficient condition for almost everywhere convergence. But it has been proved[11] that T satisfies the strong (that is, mean) ergodic theorem if and only if

$$\frac{1}{n} \sum_{\nu=0}^{n-1} |\phi^{-\nu} e| \leqq M |e|, \tag{4}$$

and that this condition (4) implies (2). *Thus* (2) *is also redundant and* (4) *alone is the single condition which is equivalent to the strong theorem and which implies the almost everywhere theorem.* In the case where ϕ is measure preserving (2) has been established by G. D. Birkhoff,[12] N. Wiener,[13] K. Yosida and S. Kakutani,[14] and H. R. Pitt[15] by proving an inequality of the type

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[10] S. Banach, *Sur la convergence presque partout des fonctionelles linéaires*, Bull. Sci. Math. (2) vol. 50 (1926) pp. 36–43. This theorem of Banach has also been used by Yosida in generalizing the ergodic theorem of G. D. Birkhoff, see K. Yosida, *Ergodic theorems of the Birkhoff-Khintchine's type*, Jap. J. Math. vol. 17 (1940) pp. 31–36.

[11] This was discovered by D. S. Miller and the author who will publish the proof elsewhere.

[12] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U. S. A. vol. 17 (1931) pp. 656–660.

[13] N. Wiener, *The ergodic theorem*, Duke Math. J. vol. 5 (1929) pp. 1–18.

[14] K. Yosida and S. Kakutani, *Birkhoff's ergodic theorem and the maximal ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 165–168.

[15] H. R. Pitt, *Some generalizations of the ergodic theorem*, Proceedings of the Cambridge Philosophical Society vol. 38 (1942) pp. 325–343.

$$|e_\alpha| \leqq \frac{1}{\alpha}\int_S f(t)dt, \tag{5}$$

where e_α is the set where $\sum_{\nu=0}^{n-1} f(\phi^\nu t)/n > \alpha$ and f is positive. While the works of these authors make it clear that an inequality similar to (5) is valid providing $|\phi^{-\nu}e| \leqq M|e|$ it is not at all clear whether or not (4) will give anything like (5). Nevertheless (4) implies (2) which is all that is needed for the almost everywhere ergodic theorem. I believe that the ergodic theorems outlined in these pages contain most of the known[15] ergodic theorems in the cyclic case.

YALE UNIVERSITY

[15] K. Yosida and S. Kakutani, *Operator theoretical treatment of Markoff's process and mean ergodic theorem*, Ann. of Math. (2) vol. 42 (1941) pp. 188–228.

ANALYSIS IN COMPLEX BANACH SPACES

ANGUS E. TAYLOR

1. **Introduction.** Abstract spaces, and Banach spaces in particular, have played a prominent rôle in recent years in connection with many problems of analysis. There has also been a notable tendency for the concepts and tools of analysis to take a place alongside the algebraic and topological notions which are characteristic of the whole subject of abstract spaces. Thus analysis becomes more algebraic, and at the same time its range is broadened.

For a great many purposes it is immaterial whether a Banach space be real or complex. It is well known that a large portion of the theory of linear operations, as developed in Banach's book (Banach [1][1]), is equally valid for complex or real spaces. There are, however, situations in which the complex number system plays a crucial rôle. The theory of analytic functions in Banach spaces is a case in point. There are two large divisions of this subject: the theory of functions of a complex variable, the values of the functions lying in a Banach space, and the theory of analytic functions of an abstract variable. Our principal concern in this paper will be the first of these two theories. A brief survey of the second theory, and references to the literature, are given in §8.

Henceforth, except as otherwise stated, we shall use the term Banach space to mean a complex Banach space. The algebraic structure of such a space is that of an additive Abelian group with the complex numbers as operators. The topology of the space is defined by a *norm*; the norm of an element x is written $\|x\|$. It has the properties of an absolute value. Then $\|x-y\|$ is the distance between x and y, and the space is assumed to be a complete metric space.

2. **Analytic functions.** The basic development of the theory of analytic functions with values in a complex Banach space E, the independent variable being a complex number, follows the pattern of classical analysis. It seems to have been pointed out first by Wiener [1] that Cauchy's integral theorem is valid in this general setting. The usual consequences, such as the Cauchy integral formula, Liouville's theorem, and the Taylor and Laurent expansions, then follow. It is only when we come to theorems that deal in some way with di-

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

vision of function values that significant differences appear.

3. **Weak, uniform, and strong analyticity.** If E is an abstract Banach space, the space of all complex linear functionals defined on E is also a Banach space, denoted by E^*. Let us consider a single-valued function $x(z)$ with values in E, where z ranges over a region D of the complex plane. If f is an element of E^*, $fx(z)$ is a numerical function of z. If $fx(z)$ is regular in D for each f in E^* we shall say that $x(z)$ is *weakly regular* in D.

THEOREM 1. *The function $x(z)$ is regular in D if and only if it is weakly regular there.*

This theorem, in a slightly more general form, is due to Nelson Dunford (Dunford [1, Theorem 76]). Instead of requiring $fx(z)$ to be regular for each f in E^*, it is sufficient to impose the requirement for each f in a *determining manifold* in E^*. A subset S of E^* is called a determining manifold if it is a closed linear subspace of E^*, and if there is a constant M such that for each x in E the least upper bound of $|f(x)|/\|f\|$, as f ranges over S, is not less than $M\|x\|$.

The theorem depends upon the following lemma.

LEMMA 1. *If T is an arbitrary range, and $x(t)$ is a function on T into E such that $fx(t)$ is bounded on T for each f in E^* (or, more generally, for each f in a determining manifold), then $\|x(t)\|$ is bounded on T.*

The proof of the lemma (Dunford [1, p. 354]) is of the category type. We shall give a new, brief proof of Theorem 1. Let us write

$$Q(z, h) = (x(z + h) - x(z))/h.$$

Suppose that $x(z)$ is weakly regular in D. Let C be a circle of radius r, with center at z, which, together with its interior, lies in D. By Cauchy's integral formula

$$fQ(z, h) - fQ(z, k) = \frac{1}{2\pi i}\int_C fx(t)\,\frac{(h - k)dt}{(t - z - h)(t - z - k)(t - z)},$$

provided that $|h|$ and $|k|$ are less than r. Now, for each f, $fx(t)$ is bounded on C; hence, by the lemma, there is a constant M such that $\|x(t)\| \leqq M$ on C. Thus $|fx(t)| \leqq M\|f\|$, and

$$|f[Q(z, h) - Q(z, k)]| \leqq \frac{M|h - k|}{(r - |h|)(r - |k|)}\|f\|.$$

But then

$$\|Q(z, h) - Q(z, k)\| \leqq \frac{M|h-k|}{(r-|h|)(r-|k|)}.$$

It follows, since E is complete, that $Q(z, h)$ approaches a limit as $h \to 0$. This proves that $x(z)$ is regular in D. That weak regularity is a consequence of ordinary (strong) regularity is, of course, trivial.

The next theorem, discovered by the writer (Taylor [6, Theorem 2.1]), is closely related to Theorem 1. Let E_1 and E_2 be Banach spaces, and let $[E_1, E_2]$ denote the Banach space of all linear transformations of E_1 into E_2. The norm $\|T\|$ of a transformation T is defined as the least upper bound of $\|Tx\|$ for all elements x of E_1 such that $\|x\| = 1$. Let A_λ be a family of linear transformations of E_1 into E_2, the parameter λ ranging over an open set D of the complex plane. We say that A_λ is *strongly* analytic and regular in D if for each x in E_1 the function $A_\lambda x$ is regular as a function with values in E_2. We say that A_λ is *uniformly* analytic and regular in D if it is regular as a function with values in $[E_1, E_2]$.

THEOREM 2. *The family A_λ is uniformly regular in D if and only if it is strongly regular there.*

The proof of this theorem may be modeled very closely after the above proof of Theorem 1. Instead of Lemma 1 we use an analogous lemma about linear transformations (cf. Banach [1, p. 80, Theorem 5]). It is a corollary of Theorems 1 and 2 that the regularity of $\phi A_\lambda x$ for each x in E_1 and each ϕ in E_2^* implies the strong, and hence the uniform, regularity of A_λ. It is to be emphasized that Theorems 1 and 2 are not true if we substitute continuity for analyticity.

4. **Concrete Banach spaces.** A space E is said to possess a denumerable basis $\{x_n\}$ if to each x in E there corresponds a unique sequence of numbers $\{u_n\}$ such that

$$x = \sum_{k=1}^{\infty} u_k x_k.$$

If we define $f_n(x) = u_n$, f_n is an element of E^* (Banach [1, p. 111]). A number of familiar spaces have bases with the property that a series $\sum_{k=1}^{\infty} a_k x_k$ converges if the norms of the partial sums $\sum_{k=1}^{n} a_k x_k$ are bounded (Dunford and Morse [1, p. 415]). A space with a basis of this sort will be said to have property (A). The function spaces L^p $(p > 1)$, and the sequence spaces l^p $(p \geqq 1)$ are examples. The space (c) of convergent sequences has a basis, but it does not have property (A).

THEOREM 3. *Let E be a space with a denumerable basis. Let $\{u_n(z)\}$ be a sequence of numerical functions of the complex variable z, each analytic and regular in a region D, and such that*

(a) *to each compact*[2] *subset S of D and each f in E^* there corresponds a constant M such that $\left|f(\sum_{k=1}^{n} u_k(z)x_k)\right| \leqq M$ if z is in S and $n=1, 2, 3, \cdots$;*

(b) *the series $\sum_{k=1}^{\infty} u_k(z)x_k$ converges if z is in D.*

Then the function $x(z)=\sum_{k=1}^{\infty} u_k(z)x_k$ is analytic and regular in D. Its derivative is $x'(z)=\sum_{k=1}^{\infty} u_k'(z)x_k$.

We first observe that if the space has property (A), condition (b) is automatically satisfied when (a) is, by virtue of Lemma 1. Because of Theorem 1, it is enough to prove that $x(z)$ is weakly regular. This, however, follows directly from (a) and (b), by Montel's theorem. We leave it for the reader to show that $x'(z)$ is given by the series indicated.

Theorem 3 furnishes a practical method for testing analyticity. For example, it shows at once that a sequence $\{u_n(z)\}$ of functions with a common region of regularity D may be regarded as an analytic function on D into l^p if the partial sums $\sum_{k=1}^{n} |u_k(z)|^p$ are uniformly bounded in each compact subset of D. For other results of the same type, see Taylor [6, Theorem 3.1].

Consider next the space C whose elements are complex functions $x(t)$ continuous on the closed interval $[a, b]$, with the norm

$$\|x\| = \max_{a \leqq t \leqq b} |x(t)|.$$

If $x(t, z)$ is a function of t and z which belongs to C for each value of z, we shall use the notation $x(\cdot, z)$ when we wish to regard $x(t, z)$ as a function of z with values in C. Bôcher [1] used the term *semi-analytic* to describe functions $x(t, z)$ which are continuous in the pair t, z and analytic as functions of z. Bôcher's results concerning such functions, insofar as the dependence upon z is concerned, may be regarded as corollaries of the abstract theory with which we are concerned, as soon as it is shown that the hypothesis of semi-analyticity is sufficient to make $x(\cdot, z)$ an analytic function of z. The following theorem proves this and slightly more: instead of continuity jointly in t and z it is enough to require boundedness.

THEOREM 4. *A function $x(t, z)$ defines a function $x(\cdot, z)$ regular in a region D if and only if*

(a) *$x(t, z)$ is continuous on $[a, b]$ for each z in D;*

[2] Compact sets, as we use the term, are closed.

(b) *$x(t, z)$ is regular in D for each t on $[a, b]$;*

(c) *to each compact set S in D there corresponds a constant M such that $|x(t, z)| \leqq M$ if t is on $[a, b]$ and z is in S.*

By employing Cauchy's integral formula much as in the proof of Theorem 1 we can show directly that the difference quotient

$$(x(t, z + h) - x(t, z))/h$$

converges to its limit as $h \to 0$, uniformly with respect to t. But this means that

$$(x(\cdot, z + h) - x(\cdot, z))/h$$

converges to a limit in the topology of C. The conditions (a)–(c) are thus seen to be sufficient. Their necessity is trivial.

The continuity of $x(t, z)$ and $\partial x/\partial z$ jointly in t and z is a consequence of the regularity of $x(\cdot, z)$.

The situation in the space L^p is much more complicated. Wiener [1] studied power series in z whose coefficients are functions in L^2. Taylor [6, Theorem 3.2] showed that a function $F(z)$ which is analytic in D, with values in L^p, can be represented by a function $x(t, z)$ which is defined when z is in D and t is on the interval $[a, b]$, but not in a certain fixed set N of measure zero. The significant thing is that this exceptional set is independent of z. The function $x(t, z)$ is regular as a function of z, and the integral $\int_a^b |x(t, z)|^p dt$ is bounded in each compact subset of D. The latter conditions are, conversely, sufficient to determine $x(\cdot, z)$ as an analytic function on D into L^p.

In the paper referred to above, Taylor studied abstract linear differential systems of the forms

$$dx/d\lambda = A_\lambda x \quad \text{and} \quad dx/d\lambda = A_\lambda x + z(\lambda),$$

where A_λ is a linear transformation depending analytically on λ, and $z(\lambda)$ is a given abstract function of λ. In concrete realizations of the abstract theory such systems include a variety of types of equations, such as integro-differential equations, differential equations in an infinity of unknowns, and so forth.

5. **Singularities.** Isolated singularities may be classified, as in classical theory, under the headings: (1) removable singularities, (2) poles, (3) essential singularities. An isolated singularity is removable if it disappears when a proper definition is given to the function at the point in question. We can prove Riemann's theorem: an isolated singularity z_0 is removable if $\|x(z)\|$ is bounded in the neighborhood of z_0. An isolated singularity z_0 is a pole if, for a suitable positive

integer m, $(z-z_0)^m x(z)$ has a removable singularity at z_0. The smallest admissible integer m is called the order of the pole. An equivalent definition may be based upon the nature of the Laurent expansion about the point z_0. By Lemma 1 and Riemann's theorem we may show that if, for each f in E^*, $fx(z)$ has a pole of order not greater than m at z_0, then $x(z)$ has a pole of order not greater than m; and if the pole is of order exactly m, then there is at least one f in E^* such that $fx(z)$ has a pole of order m at z_0.

Let us consider spaces with denumerable bases. It appears from Theorem 3 that in such a space the singularity of a function $x(z)$ may arise either from a singularity of one of the component functions $u_n(z)$ or from the behavior of the series

$$x(z) = \sum_{k=1}^{\infty} u_k(z)x_k$$

as a whole. As an example, consider the Hilbert space l^2, and the function $x(z)$ with components $u_n(z)=z^n$. Since

$$\|x(z)\|^2 = \sum_{k=1}^{\infty} |z|^{2k},$$

it is clear that $x(z)$ is regular if $|z|<1$, but not defined if $|z|\geqq 1$. The component functions, individually, give no indication of the fact that $|z|=1$ is a natural boundary.

We can prove, however, that an isolated singularity is always traceable to one of the components.

THEOREM 5. *Suppose that the conditions of Theorem 3 obtain in a region* D: $0<|z-z_0|<R$, *and suppose that each of the functions* $u_n(z)$ *is, in addition, regular at* z_0. *Then the series* $\sum_{k=1}^{\infty} u_k(z)x_k$ *converges if* $|z-z_0|<R$, *and defines a function regular, not only in* D, *but at* z_0.

To prove the theorem, consider an element f of E^*, and the series

$$fx(z) = \sum_{k=1}^{\infty} u_k(z)f(x_k).$$

The terms of the series are regular if $0\leqq|z-z_0|<R$. If $0<R_1<R$, there exists a constant M such that

$$\left|\sum_{k=1}^{n} u_k(z)f(x_k)\right| \leqq M$$

if $|z-z_0|=R_1$ and $n=1, 2, \cdots$. The same inequality must hold if

$0 \leqq |z-z_0| < R_1$, by the maximum modulus theorem. Hence, letting $n \to \infty$, we deduce that $|fx(z)| \leqq M$ if $0 < |z-z_0| \leqq R_1$. By Lemma 1 we know that $x(z)$ is bounded in the same range of z. By Riemann's theorem, then, the possible singularity at z_0 is removable. The conclusion of the theorem is now easily proved.

6. **Resolvents.** It has long been recognized that the methods of the theory of functions of a complex variable are important in the theory of linear transformations. F. Riesz, in his monograph on the subject of infinite systems of linear equations (F. Riesz [1, pp. 117–121]) indicates briefly the potentialities of the calculus of residues as a means of studying the spectrum and the characteristic manifolds of a linear transformation.

Let T be a linear transformation of the Banach space E into itself. If the transformation $T_\lambda \equiv T - \lambda I$ (I the identity) maps E into itself in a one-to-one fashion, the inverse transformation, denoted by R_λ, will also be linear. We shall call R_λ the *resolvent* of T, and the set of values of λ for which R_λ is defined the *resolvent set* $\mathfrak{R}$. All values of λ not in $\mathfrak{R}$ belong to the *spectrum* of T. A number λ is said to be in the *point spectrum* of T if it is a characteristic value, that is, if there is an $x \neq 0$ such that $Tx = \lambda x$.

Our discussion will center around R_λ as a function of λ. As far as questions of analyticity are concerned, it is immaterial whether we study R_λ or $R_\lambda x$, because of Theorem 2. We shall study R_λ directly.

THEOREM 6. *The resolvent R_λ is analytic and regular in the resolvent set (which is open, but not necessarily connected). If λ and μ are in $\mathfrak{R}$,*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu, \tag{1}$$

$$(d^n/d\lambda^n) R_\lambda = n! R_\lambda^{n+1}. \tag{2}$$

The set $\mathfrak{R}$ contains every λ such that $|\lambda| \geqq \|T\|$, and for these values

$$R_\lambda = -\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}. \tag{3}$$

If C is a contour enclosing the spectrum of T, then

$$T = -\frac{1}{2\pi i} \int_C \lambda R_\lambda d\lambda. \tag{4}$$

Formula (1) is the well known functional equation of the resolvent, familiar in the theory of integral equations. Formula (3) is the Neumann expansion of the resolvent, in abstract form. Formula (4) ap-

pears to have been overlooked until now. For the proof of (1)–(3), see Taylor [2]. To obtain (4), multiply (3) by λ and integrate, after deforming the contour, if necessary. Other references are: Stone [1, pp. 139–141]; Hildebrandt [1, pp. 195–199]; Nagumo [1]; Taylor [6, pp. 590–593]; Lorch [1].

The resolvent set is the maximal domain of regularity of R_λ. All the singular points of R_λ belong to the spectrum of T, and a point of the spectrum is a singularity of R_λ, provided that it is a limit point of the resolvent set. The resolvent R_λ is regular at infinity, and vanishes there, by (3). Hence, by Liouville's theorem, it is either identically zero (in the trivial case that the space E consists of a single element), or the spectrum of T contains at least one point.

With each simple closed contour C lying in $\mathfrak{R}$ we may associate a transformation P defined as follows:

$$P = -\frac{1}{2\pi i}\int_C R_\lambda d\lambda. \tag{5}$$

These transformations play a fundamental rôle in the study of the singularities of R_λ. Their use in studying the spectrum has recently been indicated by Lorch. We summarize the salient properties of these transformations in a theorem (Lorch [1, Theorems 5 and 6]).

THEOREM 7. *The transformation P is a projection, that is, $P^2=P$. It is permutable with T. If C_1 and C_2 are two contours in $\mathfrak{R}$, and if P_1, P_2 are the corresponding projections, $P_1P_2=P_2P_1$, and $P_1P_2=P_1$ if C_1 lies inside C_2, while $P_1P_2=0$ if C_1 lies outside C_2.*

In proving this theorem, and in many other calculations involving contour integrals, equation (1) provides a valuable key. It may be shown that $P=I$ if and only if the spectrum of T lies within C, and that $P=0$ if and only if the spectrum of T lies outside C.

A partial analysis of the structure of T and R_λ is possible with the aid of P. There is not space in this paper for a detailed account. Let us consider merely the case of an isolated point λ_0 of the spectrum of T, that is to say, an isolated singularity of R_λ. Let C be a small contour inclosing λ_0 but no other point of the spectrum. Then the Laurent expansion of R_λ is

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n, \tag{6}$$

where

$$B_n = \frac{1}{2\pi i}\int_C (\lambda - \lambda_0)^{n-1} R_\lambda d\lambda, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

and we understand that $A_n = B_{-n}$. One can show that $B_{n+1} = T_{\lambda_0} B_n$, $n = 1, 2, 3, \cdots$. Also, $B_1 = -P$, and so

$$B_n = -T_{\lambda_0}^{n-1} P, \qquad n = 1, 2, 3, \cdots. \tag{7}$$

The simplest case is that in which λ_0 is a pole of R_λ.

THEOREM 8. *If λ_0 is a pole of R_λ, it is in the point spectrum of T. If x is a corresponding characteristic element, $Px = x$. A sufficient, but not necessary, condition for λ_0 to be a pole is that P project E into a finite dimensional linear manifold.*

To prove the first statement, let m be the order of the pole. Then from (6) we infer that $B_m \neq 0$ and $B_n = 0$ if $n > m$. Let y be chosen so that $x = B_m y \neq 0$. Then $T_{\lambda_0} x = B_{m+1} y = 0$, by (7). We may also prove that $A_0 T_{\lambda_0} = I - P$. Hence we see that $Px = x$.

To prove the second part of the theorem, we define

$$S_\lambda = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n. \tag{8}$$

It is easily proved that S_λ is regular if $\lambda \neq \lambda_0$, and that $S_\lambda P = S_\lambda$. Furthermore,

$$S_\lambda T_\lambda = P. \tag{9}$$

Now suppose that the range E_0 of P is finite dimensional. Within E_0, P coincides with the identity. From the above remarks we therefore conclude that S_λ is essentially a transformation of E_0 into itself, and that it is, within E_0, the resolvent of T. Its singularity at λ_0 is therefore that of the resolvent of a finite matrix, namely, a pole.

There is a large class of transformations T for which the only singularities of R_λ, aside from $\lambda = 0$, are poles. The most familiar case is that when T is completely continuous. It is easy, in this case, to show that the only possible limit point of the spectrum is the origin. For an isolated singularity $\lambda_0 \neq 0$ it is readily proved that P is completely continuous and that its range E_0 is finite dimensional. Theorem 8 then shows that λ_0 is a pole of the resolvent. This argument has been pointed out by several people (Schauder [1, p. 193]; Hildebrandt [1, p. 198]; Nagumo [1, pp. 79–80]). An extension to transformations which are in a sense "nearly" completely continuous has been indicated in various places (Hildebrandt, loc. cit.; Nikolskij [1]; Yoshida [1, 2]).

In case the only singularities of R_λ aside from $\lambda=0$ are poles, the transformation $z=\lambda^{-1}$ carries R_λ into a meromorphic function which may be expanded by Mittag-Leffler's theorem. If the number of poles is finite, R_λ is rational, and we get an explicit partial fraction representation. In fact, since R_λ vanishes at infinity, it is equal to the sum of its principal parts at the poles $\lambda_1, \cdots, \lambda_n$. If we enclose each pole in a small contour C_j, and denote the corresponding projection (5) by P_j, we find, from (6), that

$$R_\lambda = -\sum_{j=1}^{n}\sum_{k=1}^{m_j}(\lambda-\lambda_j)^{-k}T_{\lambda_j}^{k-1}P_j, \tag{10}$$

where m_j is the order of the pole λ_j. By integrating around a contour enclosing all the poles we find the relation

$$P_1+\cdots+P_n=I. \tag{11}$$

Finally, by combining (4) and (10), we find that if $\lambda_1, \cdots, \lambda_r$ are simple poles, while $\lambda_{r+1}, \cdots, \lambda_n$ are multiple poles,

$$T=\sum_{j=1}^{r}\lambda_jP_j+T\sum_{j=r+1}^{n}P_j. \tag{12}$$

Formulas (10)–(12) generalize certain results in the theory of finite matrices (Frobenius [1]; Schwerdtfeger [1, 2, 3]). In case the poles are all simple, T is completely reducible. This is the abstract generalization of the reduction of a matrix to diagonal form.

It is of some interest to record that if E is the Hilbert space l^2, and if the resolvent is defined by an infinite matrix with elements $R_{ij}(\lambda)$, then if T is either Hermitian or unitary, the spectrum of T consists of the singular points of the functions $R_{ij}(\lambda)$, together with the limit points of such singularities. This does not hold true for an arbitrary T, however (Wintner [1, pp. 178 and 213–214]).

7. **Functions of linear transformations.** Formula (4) of §6 may be generalized as follows: Let D be a fixed simply connected open set containing the spectrum of T. Let $\mathfrak{F}$ be the class of all single-valued numerical functions $F(\lambda)$ which are holomorphic in D. This class $\mathfrak{F}$ is a commutative ring. Let us define

$$F(T)=-\frac{1}{2\pi i}\int_C F(\lambda)R_\lambda d\lambda, \tag{1}$$

where C is a contour lying in D and surrounding the spectrum of T.

If $F(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ is an entire function, formula (3) of §6 enables us to show that $F(T)=\sum_{n=0}^{\infty}a_nT^n$.

THEOREM 9. *The class of all linear transformations $F(T)$ generated by (1) is a commutative sub-ring of the ring of all linear transformations of E into itself. The mapping defined by (1) is a homomorphism.*

The proof of the theorem consists in showing that the correspondence $F(\lambda)\rightarrow F(T)$ preserves sums and products. The preservation of sums is trivial. Suppose that $F(\lambda)$ and $G(\lambda)$ are in $\mathfrak{F}$. If Γ is a contour in D, enclosing C, and λ is on C,

$$G(\lambda)=\frac{1}{2\pi i}\int_{\Gamma}\frac{G(\mu)}{\mu-\lambda}\,d\mu.$$

If we substitute the above formula into the integral

$$-\frac{1}{2\pi i}\int_{C}F(\lambda)G(\lambda)R_\lambda d\lambda,$$

reverse the order of integration, and use formula (1) of §6, the above integral around C is found to have the value $F(T)G(T)$. This proves that products are preserved by (1).

An immediate and important consequence of Theorem 9 is that if $F(\lambda)$ belongs to $\mathfrak{F}$ and has no zeros in D, then $F(T)$ has a multiplicative inverse $[F(T)]^{-1}$; the inverse corresponds to $1/F(\lambda)$, and may be written $I/F(T)$.

Formula (1) may be written in the equivalent form

$$F(T)=\frac{1}{2\pi i}\int_{C}\frac{F(\lambda)}{\lambda I-T}\,d\lambda, \tag{2}$$

by recalling that $R_\lambda=(T-\lambda I)^{-1}=-(\lambda I-T)^{-1}$. In the finite dimensional case this formula seems to be due to Poincaré, who used it in the study of continuous groups (Poincaré [1, 2, 3]). It has appeared elsewhere (Buhl [1, pp. 37–38]; Schwerdtfeger [1, p. 311]; Giorgi [1, p. 7]; Wedderburn [1, p. 130]). A special case of the formula was used by Hille [1, pp. 3 and 22] in studying semi-groups of transformations in Banach spaces. Theorem 9 seems to be new, and apparently formula (1) itself is also new in the generality with which it is presented here.

As an interesting application, we shall use the above theory to solve the linear differential equation

$$d^nx/dt^n+a_1d^{n-1}x/dt^{n-1}+\cdots+a_nx=y(t) \tag{3}$$

with the initial conditions

$$x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0. \tag{4}$$

The coefficients $a_1, \cdots, a_n$ are assumed to be constants, and $y(t)$ is a given function, continuous on the interval $[0, 1]$.

If C is the Banach space of functions $x(t)$ continuous on $[0, 1]$, let $Tx(\cdot) = y(\cdot)$ mean that

$$y(t) = \int_0^t x(s)ds. \tag{5}$$

One may show without trouble that the solution of the integral equation

$$x(t) - \lambda \int_0^t x(s)ds = y(t)$$

is

$$x(t) = -\frac{1}{\lambda} y(t) - \frac{1}{\lambda^2}\int_0^t e^{(t-s)/\lambda} y(s)ds. \tag{6}$$

The spectrum of T consists of the single point $\lambda = 0$, and the resolvent of T is defined by (6).

The differential system (3) and (4) is equivalent to the equation

$$x + a_1 Tx + \cdots + a_n T^n x = T^n y. \tag{7}$$

If $F(\lambda) = 1 + a_1\lambda + \cdots + a_n\lambda^n$, (7) becomes $F(T)x = T^n y$. Since $F(\lambda)$ and $1/F(\lambda)$ are regular at $\lambda = 0$, and hence in the neighborhood of the spectrum of T, the inverse of $F(T)$ exists, and the solution of (7) is

$$x = (T^n/F(T))y.$$

Thus, using formula (1),

$$x = -\frac{1}{2\pi i}\int_C \frac{\lambda^n}{F(\lambda)} R_\lambda y d\lambda, \tag{8}$$

where C is a contour enclosing $\lambda = 0$ and excluding the zeros of $F(\lambda)$. Since the resolvent is defined by (6), (8) gives us an explicit means of calculating $x(t)$. It is convenient to make the substitution $z = \lambda^{-1}$. We then obtain

$$x(t) = \frac{1}{2\pi i}\int_0^t y(s)ds \int_\Gamma \frac{e^{z(t-s)}}{z^n + a_1 z^{n-1} + \cdots + a_n} dz, \tag{9}$$

where Γ is a contour enclosing the roots of the polynomial $z^n + a_1 z^{n-1}$

$+ \cdots + a_n$. Formula (9) is due originally to Cauchy (see Darboux [1]).

8. **Analytic functions of an abstract variable.** In this section we shall survey briefly the theory of analytic functions where both the domain and range of the functions are Banach spaces. This theory has its roots in ideas which go back to Volterra and other pioneers in functional analysis. The most important early impetus was given by Fréchet, who in a series of papers developed the notions of functional polynomials and power series [1, 2, 4] and pointed the way to a successful calculus based upon his definition of a differential [3]. The work of Gateaux [1, 2] sketched the outlines of a theory of analytic functionals. The first systematic abstract theory of analytic functions was developed by Martin [1]. He was a pupil of A. D. Michal, under whose guidance an intensive study of the whole field of analysis and geometry in abstract spaces has been carried on for more than a decade (see the interpretation of the abstract point of view in Michal [2], and the references to the literature in Michal [1]). Various applications of the theory, in the shape given it by Martin, have appeared (Michal and Clifford [1]; Michal and Martin [1]).

An abstract theory of analytic functions, along the lines indicated by Gateaux, was announced by Graves [1], and independently by Taylor ([1 and 5]; see especially the footnote on p. 466 of [5]). In Martin's theory the basic notion is that of a power series expansion. The terms of a power series are homogeneous polynomials. For the literature on polynomials, see Fréchet [1, 4]; Martin [1]; Mazur and Orlicz [1]; Highberg [1, 2]; Taylor [3]; Banach [2]. In the theory of Graves and Taylor, $F(x)$ is said to be holomorphic in a region D if it is continuous there and if the limit

$$\delta F(x; y) = \lim_{\lambda \to 0} (F(x + \lambda y) - F(x))/\lambda$$

exists for each y in the Banach space and each x in the region D. It can be shown that $\delta F(x; y)$ is the Fréchet differential, and that $F(x)$ is analytic in the sense of Martin. The two different approaches result in the same general theory. There are certain connections between the above ideas and the work of Fantappié [1, 2] on analytic functionals. Wintner's theory of regular power series [2] is closely allied to the abstract theory described above.

One illustration of the general theory is to be found in the study of resolvents. If we regard the linear transformation T as a variable, and denote the resolvent by $R(\lambda, T)$, the resolvent is an analytic

function, not only of λ, but of T as well.[3] The Fredholm theory provides an explicit exhibition of the dependence of the resolvent upon λ and T. An abstract treatment of the Fredholm determinant and first minor was given by Michal and Martin [1]. Closely allied results, for transformations of finite norm in Hilbert space, were given by Smithies [1], though without any explicit use of a theory of abstract analytic functions. It would seem to be worth while to pursue the study of $R(\lambda, T)$ and of the functions $F(T)$ defined in §7 from this general point of view.

A theory of analytic functions from Riemann's point of view may be built up by starting from a real Banach space B, and constructing a complex Banach space in much the same way that the complex numbers are formed from the real numbers. The only part of the construction which offers the least difficulty is the defining of a suitable norm of the complex element $x+iy$, where x and y are in the real space B. That $(\|x\|^2+\|y\|^2)^{1/2}=\|x+iy\|$ will not do, in general, was pointed out in a paper by Michal, Davis, and Wyman [1]. A satisfactory definition is the following, due to Taylor (Michal and Wyman [1, p. 249]):

$$\|x+iy\| = \sup_{\|f\|=1} (|f(x)|^2+|f(y)|^2)^{1/2}, \qquad f \in B^*.$$

With the complex Banach space so constructed it is possible to develop analogues of the Cauchy-Riemann equations, and a corresponding notion of biharmonic functions (Taylor [4]). It is also possible to extend a "real" analytic function, defined by a series of homogeneous polynomials, into the complex Banach space (Taylor [3, pp. 312–315]). This process utilizes certain relations between abstract polynomials and their polars. There are various unsolved problems connected with bounds and inequalities which make the present state of this work unsatisfactory.

There is a converse problem: can every complex Banach space E be decomposed into a real and an imaginary part? The question may be put in the following form (cf. Michal and Wyman [1, Theorem 3]): does there exist a continuous transformation T of E into itself, having the properties $T^2x=x$, $T(x+y)=Tx+Ty$, $T(ax)=\bar{a}Tx$? We may call such a transformation a *conjugation*. Let us call a subset M of E a *maximal essentially real* linear manifold if

[3] In this connection we note that a function $F(\lambda, x)$ of a complex variable λ and a Banach variable x is analytic in the pair λ, x if it is analytic in each variable separately (Taylor [5, Theorem 3.2]). Whether the like is true of functions of two abstract variables is at present unknown.

(a) it is closed under addition and under multiplication by real numbers;

(b) x and ix are both in M only if $x=0$;

(c) M is not a proper subset of any set with properties (a) and (b).

Then it is not difficult to show that there exists a conjugation of E if and only if there exists in E a closed and maximal essentially real linear manifold. This whole problem has been studied by Taylor [7], but no complete solution has been reached.

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UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF.

HENRY SEELY WHITE—IN MEMORIAM

Henry Seely White, emeritus professor of mathematics at Vassar College, died in Poughkeepsie, New York, on May 20, 1943, his eighty-second birthday. In his death mathematics has lost a distinguished geometer, the college a wise and helpful guide, and his colleagues and the community a much loved friend.

Professor White was born in Cazenovia, New York, on May 20, 1861, the son of Aaron White, principal and teacher of mathematics in Cazenovia Seminary. Of English stock, his first American ancestor was John White who came to the United States in 1632, one of the original settlers of Hartford. The schools and seminary of Cazenovia gave Mr. White his early training and preparation for college. He entered Wesleyan University and graduated with honors in 1882 at the age of twenty-one. During the following year he assisted in astronomy and physics at Wesleyan and returned there again in 1884 after a year at Centenary College where he taught mathematics and chemistry. On completion of three years as registrar and tutor in mathematics in Wesleyan he went to Göttingen to work with the unusual group of mathematicians in the university at that time, and completed his doctorate there in 1891.

It was during this period that he received stimulus and training which gave shape and scope for his mathematical development, strengthening of his independence in research, and a beginning in reading in a variety of fields which is no longer such a common attribute of our great specialists. While his publications were in the field of his specialized geometry, his studies ranged over the entire field. It was this wide scope that made him such a valued adviser and guide both to his own generation and to younger mathematicians.

On his return to the United States, Professor White spent two years in the department of mathematics at Clark University and then was called to Northwestern, one of a large group of unusually able young men. It was from Northwestern that he came to Vassar, in 1905, to be head and guide of the department of mathematics until the age of 75, and close friend and frequent lecturer for the remaining years of his life. Students as well as faculty members prized each opportunity to enjoy his mind. In the week of his death he was scheduled to lecture to a class in the geometry of space curves.

In the mathematics organizations Professor White took an active part as organizer, editor and officer. He was on the editorial staff of

the Bulletin of the American Mathematical Society for a number of years, was editor of the Annals of Mathematics from 1899 to 1906, and editor of the Transactions of the American Mathematical Society from 1907 to 1914. He served the American Mathematical Society as president from 1906 to 1908 and the American Association for the Advancement of Science as vice president in 1915. In the same year he was elected Fellow of the National Academy of Sciences. Honors conferred upon him included an LL.D. from Northwestern, 1915, and D.Sc. conferred by Wesleyan at the time of its 100th anniversary in June, 1932.

Among Professor White's publications were contributions to the Bulletin of the American Mathematical Society, the American Journal of Mathematics, and a text of *Plane curves of the third order*. While his studies ranged over a large field in the subject and far outside it, his main interest was in the theory of invariants, the geometry of curves and surfaces, correspondences, algebraic plane and twisted curves, homeomorphic sets of lines in a plane, and relativity in mechanics.

Professor White's life in the community was broad and rich. He and Mrs. White had wide connections through church and city organizations, especially through the University Club, the Yacht Club, and the Vassar Institute. Many in all these varied groups mourn the loss of an acute mind, a friendly, courteous wit, and a great heart.

MARY EVELYN WELLS

BOOK REVIEWS

Analytic topology. By G. T. Whyburn. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278 pp. $4.75.

In an effort to place the subject matter of this book within the larger domain covered by the title, we take up two illustrations that are in some sense typical of the contents. The first selection is for those who prefer to think in terms of invariants. One finds many-to-one transformations each of which preserves, a priori, certain topological properties, and a subsequent search for new invariants, rather than a study of isometric or other invariants that might seem to fall within the title. Let F be a class of transformations and S be a class of spaces. It is desired to choose S so that any element of S maps onto another element of S under every mapping from F, and that, furthermore, there be in S a particular "generator" E, such that, given any element S^* of S, there is an element f in F for which $f(E)=S^*$. For the case of locally connected continua S, and the continuous mappings F, this result is well known; here E may be taken as an arc. Other examples are: boundary curves S, simple closed curve E, and non-alternating transformations F; cactoids S, sphere E, and monotone mappings F; or hemi-cactoids, 2-cell, and monotone mappings. In many instances $f(S^*)=E$ is also possible for some f in F. The composition of two elements of F is always in F, at least on spaces from S. A similar situation in which F is unknown and S contains only one element, that is, when the original and image spaces are homeomorphic, is of considerable importance and largely an open question.

The second selection serves to emphasize the highly natural relationship between the older structural ideas and the analytic concepts that form the major part of this work. In order that a continuum M should be a simple continuous arc it is necessary and sufficient that M contain at most two points that do not separate M. This characterization is due to R. L. Moore. By its use of an array of cut points between the two end points, it strongly suggests a powerful technical use of the notion of a cutting for singling out linear arrays in continua. One might say that the theories of dimension and regularity given by Menger and Urysohn are based on a similar, but not the same, approach. When the cuttings are points this technique, roughly speaking, culminates in the cyclic element theory for certain special continua. For non-separated cuttings of more than one point this technique yields other results that are unfortunately not as well

known (Chapter III). It also affords a simple and natural transition from the structural point of view to the analytic in the form of the non-alternating transformation, since this transformation has (in addition to continuity) the property that those counterimages $f^{-1}(b)$ which are cuttings form a non-separated collection. When considered in conjunction with cyclic elements this mapping acts in a way that suggests a common background with these elements. But the non-alternating transformation also has properties in common with the strongly analytic (Chapter X) interior transformation which preserves open sets. These relations become more evident when non-alternating is specialized to monotone $-f^{-1}(b)$ connected for all b in B. In this connection see quasi-monotone. Finally, as was shown by Moore, the monotone image $B=f(E)$ of a topological sphere is again a sphere provided no continuum $f^{-1}(b)$ separates E. Since the general properties of these mappings are thoroughly treated it is evident that some parts of this book provide a highly appropriate setting for certain results from the work of Moore, Menger and Urysohn. Various other selections would show a strong representation of other mathematicians.

The author, apparently motivated by a desire to select a nucleus of basic material, has left out a considerable portion of his own work, that of his students, and others, particularly work dealing with local separating points and with property S. There is no extensive use of metric methods, combinatorial topology, nor group theoretic material. After brief study the book permits considerable flexibility, but the various chapters can not be read independently.

Since the contents are self-contained and carefully integrated, this volume will have considerable interest for students and certain possibilities as a text. In this connection the various exercises in the first three chapters deserve mention; but it must also be said that it moves swiftly into the chosen course and does not contain many examples.

Roughly, the purely analytic topology occupies the latter half of the book, but one finds a growing use of analytic methods beginning early (Chapter II). In the following outline by chapters, some notices of errors or omissions are given, together with additional cross references and other comments, but these remarks are neither systematic nor complete.

The first half of Chapter I leads from a set of six axioms to the metrization theorem. It is then assumed that all spaces are separable and metric. Compactness is used in the sense of self-compact (subsets conditionally compact). The student will probably wish to read supplementary material concerning the idea of compactness. A locally

compact and connected set is a generalized continuum. The chapter continues with connectivity for sets and their limits superior, and then takes up continua, the Brouwer reduction theorem, continua of convergence, and local connectivity. The concluding section on property S and uniformly locally connected sets makes a careful comparison of these important properties. Some of the logical niceties of definition are absent, and on page 4, $q \subset K$ should be $q \in K$.

The second chapter will probably overshadow many other parts of the book as an item of general interest because of its classical results, the interesting nature of the proofs, and the way it gives emphasis to the title of the book. Using the familiar existence-extension technique for uniformly continuous mappings, Whyburn reduces the existence part to a flexible central result which is subsequently used to characterize locally connected continua as continuous images of the unit interval, and compact sets as mappings of the Cantor discontinuum. Furthermore this central result yields an imbedding theorem which, paired with arcwise connectivity for locally connected continua, gives the Moore-Menger arcwise connectivity for complete connected and locally connected sets, and other results. This imbedding theorem furnishes insight into "arcless" continua, but, due to the arrangement of the material, this comment (furnished by the original paper) was omitted here. An altered form of the central result established by Harrold is included, although many of his related results are not. The chapter begins with the necessary preliminaries on transformations, complete sets, and uniform continuity; it concludes with the arcwise connectivity theorem for continua. This final result is established in three ways, including the Kelley proof, so short that it once enjoyed considerable circulation in oral form. The symbols for closure and complete closure (dash and tilde, page 28) could be confused, and the waves over certain letters on page 30 are poorly done. Otherwise the typography is excellent through the whole book.

The fundamental importance of the treatment of non-separated cuttings in Chapter III has been indicated earlier in this review. In addition to the author's order (Menger-Urysohn) theorems for various separating points, the Moore characterizations of the arc and simple closed curve, and the theorem of W. H. Young, there is much basic material on (linear) order, Borel classes, and the nature and existence of cuttings. In particular the set $E(a, b)$ of all points separating two given points has a natural order, is closed for some continua, and—the author used the Wilder proof—compact in locally connected continua. Various examples might be given; the book has none. In line 5 from bottom of page 41, "points" should be "sets."

That the order is "natural," top of page 43, must be shown.

Chapters IV and V might well have been prefaced by the author's paper, *Concerning maximal sets* (Bull. Amer. Math. Soc. 1934), which does not appear in the bibliography. In every other sense these chapters are a polished presentation of the cyclic element theory, incorporating the recent extension to semi-locally-connected continua, the work of Ayres on cyclic connectedness and the Whyburn contributions to multicoherence (Eilenberg) and the classification of continua among curves. The results in section 7, Chapter IV, are particularly important; see also the theorems at the top of page 107. The lemma page 175, top, should be mentioned on pages 72 and 73. Those readers seeking a casual knowledge of cyclic elements may reach this end by studying dendrites (Chapter V), provided they use the pairings: cyclic element—point; cyclic chain—arc; H-set—connected subset; A-set—subcontinuum. On page 73, part (c) of (7.1), omit the vertical bar before H. The references for Chapter IV are particularly rich in worthwhile material; it would have pleased the reviewer to find the work of Kelley, Harry, and Youngs included in the text. Additional reference: Jones, *A posyndetic continua and certain boundary problems* (Amer. J. Math. 1941).

The special properties that inhere in continua, their complements, and their boundaries, when the containing space is a plane or a sphere, are developed in Chapter VI. This chapter is fundamental—see, for example, the characterization of the sphere—for readers of the book as a whole, and is second to none in its appeal to the casual reader. It is of interest to note that boundary curves take their name from the property exhibited on p. 107. Separation theorems and accessibility are covered, but all the work on primitive skew curves is omitted.

The theory of semi-continuous decompositions is developed and integrated with that of continuous transformations early in Chapter VII. Certain types of transformations are then defined and characterized. One type, $f(A)=B$, with $f^{-1}(b)$ totally disconnected, called "light" becomes increasingly important in the next few chapters. Semiclosed sets, null collections, and related factorizations constitute the main material. However, the brief treatment of the question of finding an $f(A)=B$ such that A and B are homeomorphic should not be overlooked. In this connection a marginal note on page 135 referring to page 171 will prove valuable. In Chapter VIII retractions, quasimonotone transformations and the relative distance mapping are added to the list. The k to 1 mappings are noticeably absent; the 0-regular and local homeomorphism are only briefly considered later. This chapter is nevertheless full of material from the heart of the

analytic theory. It is highlighted by the monotone-light factorization theorem, the Wallace quasi-monotone theory (middle ground between interior and monotone), some unpublished work on non-alternating mappings of simple links and A-sets, and new proofs in the work with the relative distance transformation. See the application page 206. In the references Wallace [4] should be [3].

Chapter IX deals with applications of monotone and non-alternating transformations. It presents a pleasing blend of general results and particular applications which fix and enrich the concepts which are involved. The author adds materially to the interest by pausing to discuss the general characteristics of certain theorems. The last section is entirely new and the work of Morrey and Moore on the various cactoids is treated using new proofs. These invariants together with some striking characterizations of non-alternating transformations on boundary curves form the major part of the chapter. The work of Steenrod and Roberts on the monotone images of two-dimensional manifolds is not included. Correction: last line page 181 Lemma (3.41) should be (3.31).

Chapter X is devoted exclusively to interior transformations: on linear graphs the one-dimensional Betti group maps homomorphically; on simple closed curves the mapping is like $w=z^k$ on $|z|=1$, or the image is an arc. For the light interior transformation on a compact space, there is an arc in the original set that maps topologically onto a given arc in the image space. This inversion of arcs recalls an open question. However, see the example in the book. The treatment of light interior mappings on a 2-cell virtually establishes the invariance of a 2-cell, but is conceived and used as a central tool in proving the invariance of the 2-manifold property, and the action of these mappings in the small on such manifolds. The manifolds are not required to be compact, and thus this new work further supplants pioneer results of Stoïlow. The local analysis—$w=z^n$ on $|z|\leq 1$ in one instance—is used, together with some new results on local homeomorphisms, to introduce an analysis in the large. This culminates in a simple numerical relationship between the Euler characteristics of the manifolds and leads to a considerable number of examples and applications—closed surfaces retain orientability under the inverse, spheres map onto spheres, projective planes, or 2-cells, and so on. Certain results are then extended to pseudo-manifolds using the relative distance transformation. Some outstanding new items are the proofs by direct analysis of the action of interior mappings. Open questions are: interior (non-light) action on manifolds; the special properties of the monotone factor of an interior transformation.

References to Puckett at end of the previous chapter, and his work on the inversion of local connectivity, are pertinent to this chapter. The latter is not in the bibliography.

The first part of Chapter XI deals with locally connected continua, non-alternating interior transformations and retractions, and the existence of these mappings. The second half of the chapter treats mappings onto the circle using the methods of Eilenberg. A second major type of factorization for transformations arises from the concept "equivalence to one." Each mapping of one simple closed curve into another is homotopic to an interior mapping. Results such as these help to orient the student in the latter part of this chapter. Corrections: "onto" should be "into" page 219, line 6 from the bottom; 0 should be 1 in line 2 of proof (5.1) page 221; $|t+(1-t)f(0)|$ should be a factor of the right member of the equation in (6.1) page 225, and in the line above, $f_0(X)$ should be $f_0(x)$.

The last chapter opens with theorems concerning the fixed point property and fixed cyclic elements; it continues with characterizations, necessary conditions, and other relations between pointwise almost periodic, regularly almost periodic, pointwise periodic, and periodic properties together with the newly defined Whyburn orbit. The central concepts of this chapter may be traced to many sources, but in general these developments are in the spirit of the work of Ayres. Nearly every page contains unpublished results and evidence of considerable improvement in the methods of proof. The full force of certain combinations of theorems becomes apparent only under systematic study, but the beginner should not find it difficult to progress using the individual theorems and numerous examples. There are numerous corrections. The third line in the proof of (1.11) page 240 should contain S and not X; p_{i_n} should be p_{i_k} in line 3, Lemma 2, page 244; and in line 1, §6, page 253, $\subset$ should be $=$. Three others, all on page 225, are: line 1 of (6.2), $\subset$ should be $=$; lines 8 and 9, $-n+1$ instead of $-n$; lines 12 and 15, $n+1$ instead of n. Finally, page 264, lines 14 and 25, z^n should be

$$e^{2\pi i/n}z \text{ or } z \exp\left(\frac{2\pi i}{n}\right)$$

This volume is obviously no catalogue of chance items of research. Some of its most striking general features are the new proofs, the integration of the material, and the way it stirs one's interest in innumerable allied topics without violating unity of form and flow of thought. To these virtues may be added a clear pithy style; many theorems are stated in one or two lines. References at the end of each

chapter, a large bibliography, a table of contents and an index, give flexible coverage of all items without being cumbersome. It should enjoy a long life and constant use by all who find interest in this type of work.

G. E. SCHWEIGERT

Poisson's exponential binomial limit; *Table* I—*Individual Terms*; *Table* II—*Cumulated Terms*. By E. C. Molina. New York, Van Nostrand, 1942. viii+46+ii+47 pp. $2.75.

If p is the probability of a "success" in a single trial, it is well known that the probability of x "successes" in n independent trials is given by

$$C_x^n p^x (1-p)^{n-x} \tag{1}$$

which is the $(x+1)$st term in the expansion of the binomial $[p+(1-p)]^n$. If the limit of (1) is taken as $p \to 0$ and $n \to \infty$ in such a way that $np=a$, one obtains

$$a^x e^{-a}/x! \tag{2}$$

which is the $(x+1)$st term of a distribution originally published by Poisson in 1837. This function not only arises as an approximation to the binomial term (1) for large n and small p, but also arises in other problems, as for example in the integration of the chi-square distribution.

Table I of the present book is a tabulation of values of (2) to six places of decimals for $a=0.001(0.001)0.01(0.01)0.3(0.1)15(1)100$ and $x=0(1)150$; x, of course, being carried far enough for each given value of a to cover values of (2) to six places of decimals, not all zero. Table II gives the values of $P(c, a)=\sum_{x=c}^{\infty} a^x e^{-a}/x!$ to six places of decimals for the same range of values of a and for $c=0(1)153$.

The book has been lithoprinted by Edwards Brothers and is bound with a flexible paper cover.

Various parts of the tables have appeared in earlier publications. For example, L. v. Bortkiewicz (*Das Gesetz der kleinen Zahlen*, Leipzig, 1898) published tables of (2) to four places of decimals for $a=0.1(0.1)10.0$ and $x=0(1)24$. H. E. Soper (Biometrika, vol. 10 (1914)) published a table of (2) to six places of decimals for $a=0.1(0.1)15.0$ and $x=0(1)37$, which was reprinted in Karl Pearson's *Tables for statisticians and biometricians*, Cambridge, 1914. E. C. Molina (Amer. Math. Monthly, 1913) published tables of c for $P(c, a)=0.0001$, 0.001 and 0.01; for $a=0.0001$ to 928, and similar,

but more extensive, tables were published by G. A. Campbell in the Bell System Technical Journal for January, 1923. The present tables are much more extensive than any of these earlier tables.

These tables have great value in probability and statistical problems. They have been used very extensively, for example, by mass production quality engineers in the development of sampling inspection plans. The sampling inspection tables published by Dodge and Romig in the January, 1941, issue of Bell System Technical Journal were calculated primarily from Molina's tables. Because of the frequent occurrence of the term $a^x e^{-a}/x!$ in various functions, the tables will probably also be found very useful in the tabulation of such functions.

S. S. WILKS

Non-euclidean geometry. By H. S. M. Coxeter. (Mathematical Expositions, no. 2.) University of Toronto Press, 1942. 15+281 pp. $3.25.

There seems to be a well established pattern for books on the non-euclidean geometries, according to which a more or less elaborate historical sketch is followed by a development of the foundations of the geometries. There is usually little space left available for developing the geometries much beyond the foundations. Thus it not infrequently happens that many interesting results not intimately connected with the beginnings of the subject are declared "beyond the scope of this book."

Though the plan of the book under review presents no radical departure from such a pattern it does offer somewhat more of the subject proper than is usual, while the manner in which it accomplishes its aims sets a new high standard for such texts.

The historical survey is confined to a short first chapter (which the author observes "can be omitted without impairing the main development"). The emphasis being on the projective approach, there follow three chapters concerning the foundations of real projective geometry. Chapters V, VI, and VII are devoted to elliptic geometry of one, two, and three dimensions. Introducing congruence axiomatically (IX) in a "descriptive" geometry (VIII), the author obtains an absolute geometry from which the euclidean and hyperbolic geometries follow upon adjoining the appropriate parallel axiom. Two-dimensional hyperbolic geometry is treated in Chapter X. The next three chapters, entitled Circles and Triangles, The Use of a General Triangle of Reference, and Area, deal, for the most part, with matters apart from the foundations, while the concluding chap-

ter of the book presents the usual euclidean models of the non-euclidean geometries.

The book is heartily recommended as furnishing a very well written account of the fundamental principles of hyperbolic and elliptic geometry from the classical point of view. This was doubtless the author's aim. The reviewer feels constrained to observe, however, that as a modern treatment of an old subject, the book might have recognized some of the contributions to its field made available during the last few years. Thus to the three traditional avenues of approach to the non-euclidean geometries there has been added a fourth—the abstract metric approach—which injects into the rigor of the classical axiomatic methods the stimulus of a rapid development. Still more within the spirit of the book, it seems, would have been a notice of the foundations of hyperbolic geometry due to Menger, Jenks, and Abbott based upon the sole operations of joining and intersecting.

The typography is excellent, and the figures (for the most part simple) are carefully drawn. Those contemplating using the book as a classroom text are advised that it contains no exercises and that many proofs, especially concerning material in the latter part of the book, would have to be supplied.

L. M. BLUMENTHAL

NOTES

The Navy is urgently in need of women with mathematical training for the Aerology and Radar programs which are sponsored by the WAVES. For both programs it is expected that the candidates be college graduates. For the Radar program, the candidate must have completed with at least a "C" average not less than one year of college physics and one year of college mathematics. For the Aerology program, she must have had at least one year of Calculus and one year of Physics. Interested women should apply to any office of Naval Officer Procurement.

Professor J. L. Synge, formerly of the University of Toronto now of the Ohio State University, has been awarded the Henry Marshall Tory medal of the Royal Society of Canada for achievement in scientific research.

Professor R. C. Archibald of Brown University has retired with the title emeritus.

Professor H. F. Baker of the University of Cambridge, Professor G. D. Birkhoff of Harvard University and Professor Oswald Veblen of the Institute for Advanced Study have been elected honorary fellows of the Royal Society of Edinburgh.

Professor P. W. Bridgman and Professor Harlow Shapley of Harvard University have been elected corresponding members of the Mexican National Academy of Sciences.

Professor D. R. Curtiss of Northwestern University has retired with the title emeritus. He plans to make his home in California.

Professor H. W. Kuhn of the Ohio State University, whose retirement has recently been announced, has been given the title emeritus.

Professor S. E. Rasor of the Ohio State University has retired with the title emeritus.

Assistant Professor Harriet W. Allen of Hollins College, Hollins College, Virginia, has been promoted to an associate professorship.

Assistant Professor D. B. Ames of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dr. K. J. Arnold of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor Theodore Bennett of Marietta College, Marietta, Ohio, has been promoted to a professorship.

Assistant Professor M. T. Bird of Utah State Agricultural College has been promoted to an associate professorship.

Dr. R. P. Boas has been appointed visiting lecturer at Harvard University.

Assistant Professor W. M. Borgman of Wayne University has been promoted to an associate professorship.

Assistant Professor Richard Brauer of the University of Toronto has been promoted to an associate professorship.

Assistant Professor F. J. H. Burkett of Union College has been promoted to an associate professorship.

Dr. Herbert Busemann of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Nathaniel Coburn of the University of Texas has been promoted to an assistant professorship.

Associate Professor A. H. Copeland of the University of Michigan has been promoted to a professorship.

Assistant Professor H. S. M. Coxeter of the University of Toronto has been promoted to an associate professorship.

Dr. John DeCicco of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. R. P. Dilworth of Yale University has been appointed to an assistant professorship at the California Institute of Technology.

Associate Professor Nelson Dunford of Yale University has been promoted to a professorship.

Dr. Paul Erdös of the University of Pennsylvania has been appointed visiting lecturer at Purdue University.

Assistant Professor D. G. Fulton of Ohio Northern University has been appointed to an assistant professorship at the University of New Hampshire.

Associate Professor W. H. Gage of the University of British Columbia has been promoted to a professorship.

Dr. Abe Gelbart has been appointed to an assistant professorship at Syracuse University.

Sister Agnes A. Green of the Immaculate Heart College, Los Angeles, California, has been promoted to an associate professorship.

Dr. P. R. Halmos of the University of Illinois has been appointed to an assistant professorship at Syracuse University.

Assistant Professor O. G. Harrold of the Louisiana State University has been appointed to an assistant professorship at Pomona College.

Assistant Professor G. G. Harvey of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. F. B. Hildebrand of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. A. S. Householder of the University of Chicago has been promoted to an assistant professorship of mathematical biophysics.

Assistant Professor G. B. Huff of Southern Methodist University has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan has been appointed to a professorship at the University of British Columbia.

Assistant Professor Marie M. Johnson of Oberlin College has been promoted to an associate professorship.

Dr. Mark Kac of Cornell University has been promoted to an assistant professorship.

Associate Professor Claribel Kendall of the University of Colorado has been promoted to a professorship.

Dr. J. P. LaSalle of the Massachusetts Institute of Technology has been appointed research associate at Princeton University.

Professor C. C. MacDuffee of Hunter College has been appointed to a professorship at the University of Wisconsin.

Assistant Professor M. G. Malti of Cornell University has been promoted to a professorship of electrical engineering.

Dr. Rhoda Manning of Oregon State College has been promoted to an assistant professorship.

Professor Deane Montgomery of Smith College is on leave at Princeton University.

Associate Professor C. A. Nelson of New Jersey College for Women, Rutgers University, has been promoted to a professorship.

Dr. H. E. Newell of the University of Maryland has been promoted to an assistant professorship.

Professor Abba V. Newton of Hartwick College, Oneonta, New York, has been appointed to an assistant professorship at Smith College.

Dr. Ivan Niven of Purdue University has been promoted to an assistant professorship.

Dr. E. P. Northrop of Hotchkiss School, Lakeville, Connecticut, has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Gordon Pall of McGill University has been promoted to an associate professorship.

Assistant Professor P. M. Pepper of the University of Notre Dame has been promoted to an associate professorship.

Dr. Harry Pollard of Harvard University has been appointed to an assistant professorship at Kenyon College, Gambier, Ohio.

Dr. A. E. Poole of Montana School of Mines has been appointed to an assistant professorship at the University of Oregon.

Associate Professor G. B. Price of the University of Kansas has been promoted to a professorship.

Dr. S. E. Rauch of Santa Barbara Teachers College has been appointed to an assistant professorship at the University of Oregon.

Dr. J. K. Riess of Brown University has been appointed to an assistant professorship at Tulane University.

Professor L. D. Rodabaugh of Shurtleff College, Alton, Illinois, has been appointed visiting lecturer at Oberlin College.

Dr. Raphael Salem of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. Hans Samelson of the University of Wyoming has been appointed to an assistant professorship at Syracuse University.

Dr. O. F. G. Schilling of the University of Chicago has been promoted to an assistant professorship.

Dr. Edith R. Schneckenburger of Michigan State Normal College has been promoted to an assistant professorship.

Assistant Professor A. J. Smith of the College of William and Mary has been appointed to an assistant professorship at the Montana School of Mines.

Assistant Professor Domina E. Spencer of American University, Washington, D. C., has been appointed to an assistant professorship in physics at Tufts College.

Professor C. C. Spooner of the Northern Michigan College of Education has retired.

Assistant Professor F. H. Steen of Allegheny College has been promoted to an associate professorship and has been appointed head of the department of mathematics.

Assistant Professor B. M. Stewart of Michigan State College has been appointed to an assistant professorship at Denison University.

Assistant Professor I. D. Stewart of Whitman College, Walla Walla, Washington, has been promoted to an associate professorship.

Assistant Professor A. H. Taub of the University of Washington has been promoted to an associate professorship.

Assistant Professor E. P. Vance of the University of Nevada has been appointed visiting lecturer at Oberlin College.

Assistant Professor G. C. Vedova of the University of Maryland has been appointed to a visiting associate professorship at Haverford College.

Mr. M. L. Vest of West Virginia University has been promoted to an assistant professorship.

Assistant Professor D. L. Webb of the Texas Technological College has been promoted to an associate professorship.

Dr. Alexander Weinstein of the University of Toronto has been promoted to an assistant professorship.

Assistant Professor F. M. Wood of McGill University has been promoted to an associate professorship.

Dr. H. S. Zuckerman of the University of Washington has been promoted to an assistant professorship.

The following appointments to instructorships are announced: University of Alberta: Dr. Max Wyman; Brown University: Dr. W. H. Pell; University of Buffalo: Miss Ruth A. Brendel; University of Chicago: Mr. Daniel Zelinsky; Clark College, Atlanta, Georgia: Dr. David Blackwell; Colorado College: Mr. E. J. Zadina; Columbia University: Dr. Howard Levi; Johns Hopkins University: Mr. W. L. Doyle; Harvard University: Dr. I. S. Cohen, Dr. H. A. Jordan, Dr. G. W. Mackey, Dr. W. J. Thron; Illinois Institute of Technology: Dr. Louis Garfin; Indiana University: Dr. George Whaples; Iowa State Teachers College: Miss Ethel M. Howe; University of Iowa: Mr. Edwin Halfar; Massachusetts Institute of Technology: Dr. Glynn Owens, Mr. D. L. Thomsen; University of Minnesota: Mr. Ernest Johnston; Muhlenberg College, Allentown, Pennsylvania: Mr. J. L. Howell; New Mexico State College: Mr. N. W. Wells; New York University: Mr. P. B. Norman, Mr. L. G. Peck; University of Pennsylvania: Dr. H. D. Huskey; Princeton University: Mr. L. L. Rauch; Purdue University: Mr. Paul Irick, Dr. W. P. Reid; Rice Institute: Mr. George Piranian; University of Saskatchewan: Dr. Peter Scherk; Tufts College: Mrs. Mary L. Boas; University of Virginia: Mr. C. L. Clark; University of Washington: Dr. J. M. Kingston; University of Wisconsin: Mr. R. A. Good; Yale University: Dr. S. P. Avann, Mr. William H. Durfee, Dr. C. E. Rickart.

Associate Professor J. R. Bender of the University of Idaho died March 14, 1943. He had been a member of the Society since 1923.

Dr. Guido Fubini, formerly of the University of Turin and recently of the Institute for Advanced Study and New York University, died June 6, 1943. He had been a member of the Society since 1940.

Professor Emeritus G. A. Harter of the University of Delaware died July 22, 1943. He had been a member of the Society since 1922.

Professor James McGiffert of Rensselaer Polytechnic Institute died June 18, 1943, at the age of seventy-five years. He was a member of the Society for fifteen years.

Professor Emeritus W. F. Osgood of Harvard University died July 22, 1943. He had been a member of the Society since 1895.

The editors of the Bulletin wish to correct an error which appeared on page 346 of the current volume. In the second paragraph the words "the late" should not appear before the name C. A. Noble.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

199. A. A. Albert: *Quasigroups. I.*

Associate with every quasigroup $\mathfrak{G}$ the group $\mathfrak{G}_r$ of nonsingular transformations on $\mathfrak{G}$ generated by the multiplications of $\mathfrak{G}$, and say that a multiplicative system $\mathfrak{G}'$ is isotopic to $\mathfrak{G}$ if there exist nonsingular mappings A, B, C on $\mathfrak{G}$ to $\mathfrak{G}'$ such that $xA \cdot yB = (x \cdot y)C$ for every x and y of $\mathfrak{G}$. Every quasigroup is isotopic to a loop (that is, a quasigroup with a two-sided identity element e). The normal divisors of a loop $\mathfrak{G}$ are then shown to be the subsets eW defined for normal divisors W of $\mathfrak{G}_r$, $\mathfrak{G}$ is simple if and only if the only intransitive normal divisor of $\mathfrak{G}_r$ is the identity group. All loops isotopic to simple loops are simple. A system $\mathfrak{G}$ is homotopic to $\mathfrak{G}'$ if $xA \cdot yB = (x \cdot y)C$ for equivalent mappings A, B, C on $\mathfrak{G}$ to $\mathfrak{G}'$ which may be singular. Then a loop $\mathfrak{G}$ is *homotopic* to a loop $\mathfrak{G}'$ if and only if $\mathfrak{G}$ is homomorphic to an isotope of $\mathfrak{G}'$. (Received June 7, 1943.)

200. William H. Durfee: *Congruence of quadratic forms over valuation rings.*

Let R be a complete valuation ring whose associated residue-class field has characteristic not two. An equivalent diagonal form for an arbitrary quadratic form over R is obtained, and it is shown that two such nonsingular diagonal forms are equivalent if and only if their corresponding subforms composed of those terms having the same value are equivalent. Using this the author proves for forms over R a theorem stated by Witt for fields and extended by Jones to the p-adic integers, namely, if f, g, and h are nonsingular quadratic forms such that g and h each has no variables in common with f, then $f+g$ and $f+h$ are equivalent if and only if g and h are equivalent. (Received August 2, 1943.)

201. H. L. Lee: *The sum of the kth power of polynomials of degree m in a Galois field.* Preliminary report.

Let $M = c_0x^m + c_1x^{m-1} + \cdots + c_{m-1}x + c_m$ be a polynomial in the Galois field $GF(p^n)$. If $c_0 = 1$, M is called primary and if $c_0 \neq 0$, write deg $M = m$. Let S_m^k and R_m^k denote respectively the sum of the kth power of polynomials M of degree m, and of all M of degree less than m. By the use of two functions $\psi_m(t) - F_m$, $\psi_m(t)$, which vanish when M is primary and of degree m in one case and when deg $M < m$ in the other, the sum may be made to depend on an exponent less than k. Then S_m^k and R_m^k

depend on the remainder in the division of t^k by $\psi_m(t)-F_m$ and $\psi_m(t)$ respectively. (Cf. H. L. Lee, Duke Math. J. vol. 9 (1943) pp. 277–292.) (Received July 19, 1943.)

202. W. V. Parker: *Limits to the characteristic roots of a matrix.*

Let $A=(a_{ij})$ be a square matrix of order n with elements in the field of complex numbers; and define $S_i=\sum_{j=1}^{n}|a_{ij}|$, $T_j=\sum_{i=1}^{n}|a_{ij}|$, $U_i=2|a_{ii}|-S_i$, and $V_j=2|a_{jj}|-T_j$. Let S, T be the greatest of the S_i, T_j, respectively; and let U, V be the least of the U_i, V_j, respectively. It is shown that the absolute value of each characteristic root of A is not less than the greater of the numbers U and V and is not greater than the smaller of the numbers S and T. Similar bounds are also found for the real and imaginary parts of the characteristic roots. (Received July 23, 1943.)

203. H. E. Salzer: *Table of first two hundred squares expressed as a sum of four tetrahedral numbers.*

The following empirical theorem is conjectured: Every square integer is expressible as the sum of four positive (including zero) tetrahedral numbers $(n^3-n)/6$. It has been verified by a table prepared for the first 200 squares. This empirical theorem is a partial improvement of the statement that five non-negative tetrahedrals suffice for any integer. (See F. Pollock, Proc. Roy. Soc. London Ser. A. vol. 5 (1850).) (Received June 4, 1943.)

Analysis

204. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under translations.*

Let $F[y]$ be a functional defined and Wiener summable over the space C consisting of all functions $x(t)$ continuous in $0\leqq t\leqq 1$ and vanishing at $t=0$. In addition, let F be continuous and let it be bounded over every bounded set $x(\cdot)$ of C. (F is called continuous if $F[y^{(n)}]\to F[y^{(0)}]$ whenever $y^{(n)}(t)\to y^{(0)}(t)$ uniformly in $0\leqq t\leqq 1$, and F is bounded over every bounded set $x(\cdot)$ of C if for every positive constant B there exists a constant $K=K_B$ such that $|F[y]|\leqq K$ for all $y(\cdot)$ of C for which $|y(t)|\leqq B$, $0\leqq t\leqq 1$.) Under these conditions on the functional F the authors obtain a transformation formula for Wiener integrals under translations of the form $y(t)=x(t)+x_0(t)$ where $x_0(t)$ is a given function of C with a first derivative $x_0'(t)$ of bounded variation in $0\leqq t\leqq 1$. The transformation formula is $\int_C^w F[y]d_w y=\int_C^w F[x+x_0]\exp\{-\int_0^1[x_0'(t)]^2dt-2\int_0^1 x_0'(t)dx(t)\}d_w x$. The formula forms a basis for the calculation of various types of Wiener integrals. (Received July 30, 1943.)

205. M. M. Day: *Uniform convexity.* IV.

In this paper relationships between uniform convexity, factor spaces, and conjugate spaces are discussed. Theorem 1: A normed vector space B is uniformly convex if and only if all the two dimensional factor spaces of B are uniformly convex with a common modulus of convexity. The concept of uniform flattening is suggested by a description of a "sharp edge" on the unit sphere in terms of the norm of the space. It is shown [Theorem 2] that this is dual to uniform convexity; that is, $B[B^*]$ is uniformly flattened if and only if $B^*[B]$ is uniformly convex. It follows that a complete uniformly flattened B is reflexive. The proof of Theorem 2 uses a computation for

two dimensional spaces of the flattening in B in terms of the convexity in B^* and vice versa; Theorem 1 is then applied to complete the proof. (Received June 16, 1943.)

206. G. M. Ewing: *Existence theorems for multiple integral variation problems.* Preliminary report.

This is a sequel to a previous paper by the author (abstract 49-1-30) shortly to appear in the Duke Math. J. The methods used there for the single integral case are extended to obtain existence theorems under fairly general hypotheses for the problem of minimizing a certain k-tuple integral, $\int f(x, X)du$, on a class of continuous k-surfaces in euclidean n-space. Metric methods are used to obtain a convergent minimizing sequence but the final theorems closely resemble the Tonelli pattern. (Received July 23, 1943.)

207. Tomlinson Fort: *The weighted vibrating string and its limit.*

In this paper the problem of the vibrating string weighted with discrete particles is studied following lectures given by Bôcher years ago at Harvard. In addition to an outline of the work of Bôcher, an interesting special case is studied. A careful passage to the limit is carried through, thus obtaining the solution of the problem of the vibrating nonhomogeneous string in terms of Sturm-Liouville functions. It is believed that not only is the method new but the final form somewhat different from any previous presentation. Incidentally, under proper restrictions, the solution of a differential equation system is proved the limit of the solution of a recurrence system. The method of proof is the method of successive approximations. (Received August 4, 1943.)

208. Szolem Mandelbrojt: *Quasi-analyticity and analytic continuation. A general principle.*

The author proves a theorem which leads to results in different branches of the theory of functions. Results on quasi-analyticity on one hand, and results on the distribution of the singularities of a Dirichlet series and the distribution of the values taken by its analytic continuation, on the other hand, may be considered as particular cases of this theorem. (Received July 29, 1943.)

209. D. S. Miller: *On Carathéodory and Gillespie linear measure.*

In this paper the relation between the Carathéodory linear measure $L(A)$ and the Gillespie linear measure $G(A)$ of a plane measurable point set will be considered. The inequality $L(A) \leqq G(A) \leqq (\pi/2)L(A)$ will be shown to be the best possible by presenting sets E and F such that $L(E) = G(E)$ and $G(F) = (\pi/2)L(F)$. (Received August 2, 1943.)

210. P. C. Rosenbloom: *Sequences of polynomials, especially sections of power series.*

The domain of boundedness B of a sequence of polynomials is defined as the set of points z such that the sequence is uniformly bounded in some neighborhood of z. Relations are obtained between B, the gap-like structure of the polynomials, and the distribution of their zeros. The connection between the moduli of the zeros and

the coefficients is studied in detail. It is proved that under certain rather general conditions the arguments of the zeros are everywhere dense. If $f(z)=a_0+a_1z+\cdots$, then the order of magnitude of the zeros of $s_n(z)$ is roughly $r_n=|a_n|^{-1/n}$ for a certain subsequence of n's. The arguments of the zeros are everywhere dense except if $f(z)$ is an entire function of zero order. Fairly complete results for the case of a finite or zero radius of convergence, or of an entire function of infinite order, are obtained. For entire functions of finite positive order the author obtains essentially all the results announced by Carlson (C. R. Acad. Sci. Paris vol. 179 (1924) pp. 1583–1585), whose proofs have never been published so far as is known to the author. A connection between the zeros of $s_n(z)$ and the conformal mapping $w=z^{-n}f(r_nz)$ is also obtained. These results are applied to special power series. (Received July 31, 1943.)

211. Raphael Salem: *A singularity of the Fourier series of continuous functions.*

The following theorems are proved: I. There exists a continuous function $f(x)$ of period 2π whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at a point. II. There exists a continuous function $f(x)$ of period 2π whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at an everywhere dense set of points having the power of the continuum. (Received July 29, 1943.)

212. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* I.

The author outlines a heuristic method of detecting functional equations (mainly of the iterative compositional type) whose solutions involve arbitrary functions, by means of generalization of linear functions of variables which are solutions and which contain arbitrary constants as coefficients. It is assumed hypothetically that if a functional equation has a solution of the indicated type then the solution persists if that part of the linear function containing arbitrary constants is suitably replaced by arbitrary functions of the corresponding variables. The method indicated is heuristic in the sense defined in the author's article, *Revue de métaphysique et de morale*, 1914 (Le principe de continuation). If the method does not succeed, other functional equations frequently result whose solutions approach the solution of the original equation. The preceding effects a gradual transition from functional equations previously solved by the author by differentiation (such as certain of his quasi-transitive equations) to equations at least not readily solved in the latter manner. The preceding type of solutions is contrasted with solutions of equations obtained analogously by generalizing linear functions of particular solutions containing arbitrary constants as coefficients. Reference is made to papers previously reported in this Bulletin (vol. 23 (1917) pp. 300, 393; vol. 24 (1918) p. 279). (Received July 31, 1943.)

213. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* II.

The author applies the method of the preceding paper to the solution of functional equations previously defined by him and equations relevant to number systems (such as distributive and generalized associative equations) and in suitable instances, their formal inverses relatively to equations of the type: $\phi[f(x, y_1, y_2, \cdots, y_n), y_1, y_2, \cdots,$

$y_n]=x$. Additional hypotheses are of the type: If $f(x_1, x_2, \cdots, x_{n+1})$ is a solution, then $\psi^{-1}f[\psi(x_1), \psi(x_2), \cdots, \psi(x_{n+1})]$ is also a solution where $\psi(x)$ is an arbitrary function with an inverse. Example: $\zeta(u, y_1, y_2, \cdots, y_n)=\tau(v_1, v_2, \cdots, v_{n+1})$ where $u=\tau(x_1, x_2, \cdots, x_{n+1})$ and $v_i=\zeta(x_i, y_1, y_2, \cdots, y_n)$. Solution: $\zeta(x_1, x_2, \cdots, x_{n+1})=x_1 \cdot \alpha(x_2, x_3, \cdots, x_{n+1})$ and $\tau(x_1, x_2, \cdots, x_{n+1})=\beta(w_1, w_2, \cdots, w_n)\cdot x_{n+1}$ where $w_i=x_i/x_{n+1}$ and α and β are arbitrary functions. The preceding equation is abstractly self-inverse. Other examples with solutions are given. The solutions indicated are of course not exhaustive. Application is made in suitable instances to the domain of abstract groups; for instance, for $n=1$ the preceding equation has the solution: $\zeta(x_1, x_2)=x_1\cdot x_2^p$, $\tau(x_1, x_2)=(x_1\cdot x_2^{-1})^q\cdot x_2$ where p and q are arbitrary integers; and analogously for $n>1$. Reference is made to papers previously reported in this Bulletin (vol. 25 (1919) pp. 250, 257). (Received July 31, 1943.)

214. Andrew Sobczyk: *On the extension of linear transformations.*

By the Hahn-Banach theorem, any linear transformation T on a subspace X of a normed linear space Z, having a one-dimensional range, may be extended to the whole space with preservation of the norm. It is shown that the same is true for any linear transformation, provided that the range Y may be enlarged to a normed linear space $W\supset Y$. Specifically, a norm-preserving extension is always possible with range a subspace of the quotient space Z/X', where $X'\subset X$ is the zero subspace of T, and the norm in Z/X' is an extension of the norm on $Y\cong X/X'$ with respect to the Banach norm $\inf_{x'\in X'}\|z+x'\|$. Results are obtained on the question of when the existence of an extension requires the existence of a projection. A classification of linear transformations into four fundamental types is given, with examples showing the existence of each type. (Received July 29, 1943.)

215. Otto Szász: *On uniform convergence of Fourier series.*

The main object of this paper is to establish conditions for uniform convergence of a Fourier series at a given point. A necessary condition is that the function be continuous at that point; the additional condition restricts the coefficients to a certain type of slow oscillation, such as were used in Tauberian problems. The result yields Gibbs' phenomenon for such series at points of discontinuity. A theorem of a converse nature is: If $f(\theta)\sim\sum b_n \sin n\theta$ is continuous at $\theta=0$, and if for some positive constants K and C, $0\leqq(n+1)b_{n+1}+K\leqq(1+C/n)(nb_n+K)$, $n\geqq 1$; then $nb_n\to 0$. (Received July 26, 1943.)

216. H. S. Wall and Marion D. Wetzel: *Positive definite forms and convergence theorems for continued fractions.*

The form (1) $\sum_{p=1}^{n}(\beta_p+y)x_p^2-2\sum_{p=1}^{n-1}\alpha_p x_p x_{p+1}$ is positive definite for all $y>0$ if and only if (2) $\beta_p\geqq 0$, $\alpha_p^2=\beta_p\beta_{p+1}(1-g_{p-1})g_p$, $0\leqq g_r\leqq 1$. The J-fraction (3) $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$ is called positive definite if $\beta_p=\Im(b_p)$, $\alpha_p^2=\Im(a_p)$ satisfy (2). There exists a nest of circles $K_p(z)\supset K_{p+1}(z)$ such that the pth approximant of (3) is upon $K_p(z)$, $(\Im(z)>0)$. The distinction between the two cases $r_p(z)\to 0$, $r_p(z)\to r(z)>0$ ($r_p(z)$ = radius of $K_p(z)$) is invariant under change in the value of z in $\Im(z)>0$. If $\liminf|a_p|$ is finite, then $r_p(z)\to 0$. This, along with (2), leads to the theorem that the continued fraction (4) $1/1+c_1/1+c_2/1+\cdots$ converges if $\liminf|c_p|<\infty$ and $|c_p|-\Re(c_p)\cos(\phi_p+\phi_{p+1})+\Im(c_p)\sin(\phi_p+\phi_{p+1})\leqq[2\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p]/$

$[(1+y \sec \phi_p)\ (1+y \sec \phi_{p+1})]$, $y>0$, $-(\pi/2)<\phi_p<+(\pi/2)$, $0 \leqq g_p \leqq 1$. Also, (4) converges if $\liminf |c_p| < \infty$ and $\sum [|c_p| - \Re(c_p)] < 2$. (Received July 2, 1943.)

217. F. T. Wang: *On Riesz summability of Fourier series.* III.

Denote by $\phi_\alpha(t)$ the fractional integral of order α of an even integrable function $\phi(t)$, periodic, with period 2π. The following two results are proved: (i) If $\phi_\alpha(t)=o(t^\gamma)$, $\gamma>\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, e^{n^{1-\alpha/\gamma}}, \gamma+\delta)$, $\delta>0$, at $t=0$. (ii) If $\phi_\alpha(t)=o(t^\alpha(\log t)^{-1})$, $\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, n^{(\log n)^{1/\alpha}}, \alpha+1)$ at $t=0$. (Received June 11, 1943.)

Applied Mathematics

218. Stefan Bergman: *Solutions of linear partial differential equations of fourth order.*

Generalizing the known formula $u=\mathrm{Re}[f_1(z)+\bar{z}f_2(z)]$, $z=x+iy$, $\bar{z}=x-iy$, for the biharmonic functions (that is, functions u for which $\Delta\Delta u=16u_{z\bar{z}z\bar{z}}=0$), the author proves that for every equation $L(u)=U_{z\bar{z}z\bar{z}}+a_1u_{z\bar{z}z}+a_2u_{z\bar{z}\bar{z}}+a_3u_{zz\bar{z}}+a_4u_{zz}+a_5u_{z\bar{z}}+a_6u_z+a_7u_{\bar{z}}+a_8u=0$ where $a_\nu=a_\nu(z,\bar{z})$ are analytic functions of z and $\bar{z}$ there exist two functions of z, $\bar{z}$ and a real variable t, $\mathrm{E}_k(z,\bar{z},t)$, $k=1,2$, such that every solution of $L(u)=0$ which is regular in a star domain $\mathcal{D}$ can be represented in $\mathcal{D}$ in the form $u(\bar{z})=\mathrm{Re}\{\int_{-1}^{+1}[\sum_{k=1}^{2}\mathrm{E}_k(z,\bar{z},t)f_k \cdot (z(1-t^2)/2)dt/(1-t^2)^{1/2}]\}$. The methods and results of the paper *Linear operators in the theory of partial differential equations* (Trans. Amer. Math. Soc. vol. 53 pp. 130–155) can be applied to the functions u, satisfying $L(u)=0$. (Received July 30, 1943.)

219. G. E. Forsythe: *Note on equivalent potential temperature.*

Let a parcel of moist air be saturated with $w_s=w_s(p, T)$ tons of water vapor per ton of dry air. Let the following quantities be measured in meter-ton-second-absolute degree mechanical units: T=temperature; p=total air pressure; $L=L(T)$=latent heat of evaporation of water; $c_p=c_p(T)[c_v=c_v(T)]$=specific heat of dry air at constant pressure [volume]; $e_s=e_s(T)$=vapor pressure of saturated water vapor; $p_d=p-e_s$; $k=(c_p-c_v)/c_p$; $\theta_d=T(100/p_d)^k$. Let $\lambda=\lambda(p, T)=Lw_s/(c_pT)$. Rossby (Massachusetts Institute of Technology Meteorological Papers, vol. I, no. 3 (1932)) defines the equivalent potential temperature θ_e by the relation $\theta_e=\theta_d \exp \lambda$. He asserts without proof that, as $p\to 0$ in a process for which θ_e is constant, one has $\theta_d\to\theta_e$. The present note uses elementary estimates to prove Rossby's assertion: (i) under the oversimplifying assumption that L and c_p are bounded away from 0, as $T\to 0$; (ii) under weaker but physically artificial assumptions about L and c_p. The note includes a further discussion of the important meteorological quantity λ. (Received July 26, 1943.)

220. S. H. Gould: *The Rayleigh-Ritz method for higher eigenvalues.*

An elementary proof is given of the following theorem, fundamental in applications of the Rayleigh-Ritz method for calculating the eigenvalues of a given variational eigenvalue problem: the n roots, arranged in order of magnitude, of the determinantal equation obtained by using n coordinate functions are upper bounds respectively for the first n eigenvalues of the original problem. (Received July 14, 1943.)

221. G. E. Hay and Willy Prager: *On plane rigid frames containing curved members.*

In a recent paper (Quarterly of Applied Mathematics vol. 1 (1943) pp. 49–60), the authors have extended the method of the conjugate beam to the case of plane rigid frames consisting of straight members and loaded perpendicularly to the plane of the frame. In the present paper, the method is again extended to include frames containing curved members, with loading both in the plane of the frame and perpendicular to it. There is introduced a conjugate frame similar in shape to the original frame, but with different supports in general. This conjugate frame is subjected to a load which is a certain function of the distribution in the original frame of bending moment, Young's modulus, and the moment of inertia of the cross section. The equilibrium conditions of this conjugate frame afford a simple determination of redundant reactions in the original frame. Also, the distribution of bending moment and shear in the conjugate frame are equal respectively to the distribution of displacement and rotation in the original frame, which fact affords a simple determination of the distribution of displacement and rotation in the original frame. (Received July 31, 1943.)

222. F. B. Hildebrand: *On the stress distribution in cantilever beams.*

The problem of determining the stress distribution in a flanged cantilever beam of narrow rectangular cross section, supported in such a way that no distortion of the cross section takes place at one end of the beam, is treated as a problem in the theory of plane stress. A limiting case of anisotropy is considered, in which the material of the beam is perfectly rigid in the direction of the prescribed transverse loading which acts in the plane of the web plate. The exact solution is found in terms of a rapidly convergent series of almost periodic functions and a corresponding approximate solution is obtained in closed form by a direct application of the calculus of variations. The results are evaluated numerically in representative cases and, on the basis of comparisons with earlier solutions of related problems (F. B. Hildebrand and E. Reissner, Journal of Applied Mechanics vol. 9 (1942) pp. A-108–A-116), conclusions regarding the validity of certain assumptions and approximations are drawn. (Received July 27, 1943.)

223. E. G. Kogbetliantz: *Detailed quantitative interpretation of maps of gravitational anomalies with the aid of mathematical analysis.* Preliminary report.

The importance of gravific methods in geophysical surveying, especially in the prospecting for new oil fields, is very well known. The interpretation of gravity and other gravific maps is today limited to only qualitative conclusions. All attempts made for their qualitative interpretation, that is, for the numerical determination of important geological parameters such as depth below the surface, thickness of the disturbing layer, slope of an anticlinal flank, were unsuccessful. The reason for these systematic failures is the fact that the observed values, mapped after all usual corrections—correction for the regional anomaly included—reflect not only the effect due to the studied underground structure or body but also the gravific action of the immediate neighborhood under the station. This punctual anomaly due to the non-

homogeneity of the ground and affecting only one point-station cannot be evaluated and corrected. To eliminate from the interpretation the errors due to these unknown punctual anomalies, it is necessary to use in the computations of geological parameters only the average values. In these average values are melted all observed values and the effect of punctual anomalies is eliminated because their average values are negligible. The importance of this new interpretation-method is illustrated by an anomaly map due to an anticlinal for which the numerical values of all its geological parameters are expressed by definite integrals. These integrals are easily computed with the aid of a planimeter if the gravific map is given. (Received August 2, 1943.)

224. A. N. Lowan and H. E. Salzer: *Table of coefficients for inverse interpolation.*

Although many tables have been published to facilitate direct interpolation, there has been to date no table of coefficients to handle the more cumbersome task of inverse interpolation. To meet this need, the Mathematical Tables Project has computed the polynomial coefficients of the ratios of differences of various orders in the formula obtained by the inversion of the Everett formula for direct interpolation (H. T. Davis, *Tables of the higher mathematical functions*, vol. 1, pp. 82–83). The coefficients of the five fourth order terms were calculated to ten decimal places at intervals of 0.001 of the argument $m=(u-u_0)/(u_1-u_0)$. Also a short table gives the values of the coefficients of the ten sixth order terms. The values of the coefficients were computed at intervals of 0.1. It was not necessary to compute the coefficients of the second order terms, since these coefficients are tabulated at intervals of 0.0001 in the table of *Lagrangian interpolation coefficients* to be published by the Columbia University Press. The table here described will be of particular value whenever inverse interpolation is to be carried out in a table with a fairly large tabular interval. It is proposed to designate the coefficients here discussed as "inversolants." (Received June 11, 1943.)

225. Morris Marden: *On the stream function of axially symmetric flows.*

In his papers (Math. Zeit. vol. 27 (1926) p. 641 and Math. Ann. vol. 99 (1928) p. 629) Bergman studies the behavior of harmonic functions in the neighborhood of point and line singularities. Using the fact that the functions $\Phi=\Phi_1+i\Phi_2=(1/2\pi)\cdot\int_0^{2\pi}f(u)dt$ and $\Psi=\Psi_1+i\Psi_2=(1/2\pi)\int_0^{2\pi}f(u)(x-u)dt$, where $u=x+iy\cos t+iz\sin t$, are the potential and stream function of two three-dimensional axial-symmetric flows of an incompressible fluid, the author investigates the behavior of the stream function in the neighborhood of the above mentioned singularities. The author constructs certain functions possessing infinitely many line singularities (periodic stream functions) and proves certain properties of these functions. (Received July 30, 1943.)

226. Seymour Sherman: *Finite aspect ratio effect in the Glauert-Prandtl regime.*

The usual simplified development of the Prandtl integral equation for the circulation about a wing of finite aspect ratio is not readily extended to the case of compressible flow because the expression for the downwash due to a vortex line becomes more complicated. By first making the usual assumptions of thin airfoil theory and linearising

the equations of compressible flow and then appealing to the work of E. Reissner (*The equations of lifting strip theory for tapered wings*, Curtiss Research Laboratory Report SB-76-S-1) on the flutter of finite aspect ratio it is shown that the compressibility effect can be included in the Prandtl equation by substituting for $\partial C_{L\ inc}(y)/\partial\alpha$ the quantity $(1/(1-\beta^2)^{1/2})\partial C_{L\ inc}(y)/\partial\alpha$. This is done by setting up the problem as a boundary value problem for the appropriate elliptic partial differential equation. (Received July 26, 1943.)

227. Alexander Weinstein: *On the bending of a clamped plate.*

Let $\{p_k(x, y)\}$ denote a complete, but not necessarily orthogonal, sequence of harmonic functions of integrable square in a domain S with the boundary C. Consider the equation $\Delta\Delta w=f$ with the boundary conditions $w=dw/dn=0$. Let $w_m=GGf+\sum_{i=1}^{m}a_{mi}Gp_i$, where Gf is the Green's potential of f, and the constants a_{mi} are determined by the equations $\sum_{i=1}^{m}a_{mi}(p_i, p_k)=-(f, Gp_k)$, $k=1, 2, \cdots, m$. It is proved in this paper that w_m converges uniformly to w in $S+C$ and that $\partial w_m/\partial x$, $\partial w_m/\partial y$ converge uniformly to $\partial w/\partial x$, $\partial w/\partial y$ in every domain interior to S. The approximating functions w_m can be computed explicitly in the case when S is a rectangle. The proofs are based on the results of a paper by N. Aronszajn and A. Weinstein, Amer. J. Math. vol. 44 (1942) p. 625. (Received July 10, 1943.)

GEOMETRY

228. Herbert Busemann: *On spaces in which two points determine a geodesic.*

Let R be a finitely compact, convex, and locally strictly externally convex, metric space. A geodesic in R is defined as a continuous curve which is locally isometric with the real axis. (A symmetric variational problem in parametric form will satisfy these conditions, when the extremals are considered as geodesics). There is at least one geodesic through any two distinct points of R. If this geodesic is unique, then R is either simply connected and all geodesics are isometric with a euclidean straight line, or R has a two-sheeted universal covering space and all geodesics of R are congruent to one euclidean circle. (Received July 26, 1943.)

229. John DeCicco: *Kasner's pseudo-angle.*

The theory of functions of a single complex variable is identical with plane conformal geometry. This is not the case in the theory of functions of two or more complex variables. A set of n functions of n complex variables induces a correspondence between the points of a real $2n$-dimensional space R_{2n}. The infinite group G of these correspondences has been termed the pseudo-conformal group by Kasner. In 1908, Kasner showed that for $n=2$ a transformation of R_4 is pseudo-conformal if and only if it preserves the pseudo-angle between a curve and a three-dimensional variety. This pseudo-angle theorem can be carried over to $2n$ dimensions almost without change. A system of $(2n-1)$-dimensional hypersurfaces is said to be bi-isothermal if it is pseudo-conformally equivalent to ∞^1 parallel $(2n-1)$-dimensional flats. Any system of $(2n-1)$-dimensional hypersurfaces is bi-isothermal if and only if the pseudo-angle between the given hypersurfaces and any system of parallel lines is a biharmonic

function. Many other results are obtained by using Kasner's pseudo-angle. (Received July 27, 1943.)

230. V. G. Grove: *A general theory of surfaces and conjugate nets.*

This paper studies the projective differential geometry of surfaces and conjugate nets by means of a tensor analysis based on a connection appearing naturally in the theory. All congruences harmonic or conjugate to a surface are found. A simple geometric construction for reciprocal congruences is given independent of the quadrics of Darboux. A simple tensor method of studying conjugate nets is used. Among other applications, a certain interesting involution on each line protruding from the surface is studied. Finally some of the results are specialized to the metric geometry of surfaces in euclidian space of three dimensions. (Received July 23, 1943.)

231. C. C. Hsiung: *An invariant of intersection of two surfaces.*

Let two surfaces S_1, S_2 in ordinary space intersect at an ordinary point O with distinct tangent planes τ_1, τ_2; assume that the common tangent is distinct from the asymptotic tangents of the surfaces S_1, S_2. The author proves the existence of a projective invariant determined by the second order terms of the surfaces S_1, S_2 at the point O, and he gives the invariant a projective as well as a metric characterization. The metric characterization takes the following simple form. Let K_1, K_2 be the total curvatures of the surfaces S_1, S_2 at the point O; and R_1, R_2 the radii of curvature at O of the curves in which the tangent planes τ_2, τ_1 intersect the surfaces S_1, S_2 respectively. Then $R_2^4K_2/R_1^4K_1$ is the projective invariant under consideration. (Received July 1, 1943.)

232. C. C. Hsiung: *Projective invariants of some particular pairs of space curves.*

It is known that if two plane curves have contact of order $k-1$ at a point P, and if nonhomogeneous projective coordinates are chosen so that the expansions representing these two curves in the neighborhood of P are, respectively, $y=ax^k+\cdots$, $y=\alpha x^k+\cdots$, $a\neq\alpha$, then the ratio a/α is a projective invariant studied for $k=2$ by H. J. S. Smith (Proc. London Math. Soc. (1) vol. 2 (1867) pp. 196–248) and R. Mehmke (Schlömilchs Zeitschrift für Mathematik und Physik vol. 36 (1891) pp. 56–60, 206–213). Their results were interpreted and extended by C. Segre (Rendiconti dei Lincei (5) vol. 6 (1897) pp. 168–175 and vol. 33 (1924) pp. 325–329). The present paper supplements these investigations by studying pairs of curves in 3-space: (a) tangent at a general point with distinct osculating planes, (b) intersecting at a general point with distinct osculating planes, (c) intersecting at a general point with common osculating plane. The existence of a projective invariant determined by the third order terms is shown for each of these cases; these invariants are characterized geometrically in terms of certain double ratios. (Received June 11, 1943.)

233. C. C. Hsiung: *Theory of intersection of two plane curves.*

This paper is concerned with the study of the projective differential geometry of two plane curves intersecting at an ordinary point. The method used is similar to the

one given by the author in a previous paper (*Projective differential geometry of a pair of plane curves*, to appear in Duke Math. J.). Here two projective invariants determined by the fourth order terms in the Taylor expansions of the functions representing the two curves, at the point of intersection, are obtained and characterized geometrically. Further, on the basis of the vanishing or nonvanishing of these two invariants, the author arrives at four different types of canonical representation for the two curves at the point of intersection; the absolute invariants in the expansions of each type are interpreted geometrically in terms of certain double ratios. (Received June 11, 1943.)

234. Edward Kasner: *Motion in a resisting medium.*

Consider the motion of a particle moving in the plane under a generalized field of force and influenced by a resisting medium, the resistance R acting in the direction of the motion and varying as some function of the position (x, y), the direction y', and the speed v of the particle. Through a given lineal element E, there pass ∞^1 trajectories. If the osculating parabolas are constructed to these trajectories at E, the locus of the foci varies in shape with the nature of the resistance R. If the focal locus is a circle through the point of E, it is found that R must be of the form $A(x, y, y')v^2 + B(x, y, y')$. In the final part of the paper, this result is extended to space. Construct the osculating spheres at E to the ∞^1 trajectories passing through the lineal element E. If the locus of the centers of these spheres is a straight line, the resistance R is of the form $A(x, y, z, y', z')v^2 + B(x, y, z, y', z')$. Finally it is shown that if a single trajectory is known in the field of gravity with a resisting medium $R(v)$, then the law of resistance $R(v)$ can be completely determined. (Received July 27, 1943.)

235. Edward Kasner and John DeCicco: *A generalized theory of contact transformations.*

The authors present a generalized theory of contact transformations in the plane. Any union-preserving transformation of differential elements of order n into lineal elements is obtained by considering the osculating (up to and including contact of order n) of a given parameterized family of ∞^{n+1} curves to an arbitrary curve. If a union-preserving transformation T is such that any two unions which possess $n \geqq 2$ as the order of contact are converted by T into two unions which have at least second order contact, then T must be a contact transformation between lineal elements. As a consequence of this work, the authors find a general theory of evolutes and involutes which contains the osculating circle theory of Huygens and Bernoulli as a special case. (Received July 27, 1943.)

Logic and Foundations

236. Theodore Hailperin: *A set of axioms for logic.*

Two well known logical systems claiming adequacy for mathematics are currently studied. These are appellatively described as "Principia Mathematica" and "set-theory." A third and stronger system, called "New Foundations," has been proposed by W. V. Quine. (This system is not to be confused with his *Mathematical logic*, 1940.) Quine's system uses the usual logical primitives for the propositional calculus and the theory of quantifiers, and class membership, but makes no restrictions on the

ranges of the variables. To avoid the contradictions, Quine has a metalogical axiom to the effect that only "stratified" formulas determine classes. The purpose of the present paper is the presentation of nine formal axioms and a demonstration of their equivalence with Quine's metalogical axiom in the presence of the restricted predicate calculus and the axiom for identity. (Received July 29, 1943.)

Numerical Computation

237. H. E. Salzer: *Coefficients for numerical integration with central differences.*

The numbers $M_{2s}=B_{2s}^{(2s)}(s)/(2s)!$, where $B_{2s}^{(2s)}(s)$ denotes the $(2s)$th Bernoulli polynomial of order $2s$ for argument equal to s, were computed for $s=1, 2, \cdots, 10$, the result being exact values in lowest terms. Then these quantities were all checked by a cumulative recursion formula. (Previous calculations went only as far as $s=4$.) The numbers M_{2s} are coefficients of central differences of order $2s-1$ in the well known formula for numerical integration, sometimes called the Gauss-Encke formula or the second Gaussian summation formula. All important formulas for and references to these coefficients are included. This calculation was performed in the course of work done for the Mathematical Tables Project, National Bureau of Standards. (Received July 7, 1943.)

Statistics and Probability

238. E. J. Gumbel: *On the plotting of statistical observations.*

It is well known that there exist two stepfunctions corresponding to a continuous variate. We may attribute to the mth observation the ranks m or $m-1$. To obtain one and only one serial number m, which will, in general, not be integer, we attribute to x_m an adjusted frequency $m-\Delta$, namely the probability of the most probable mth value. The correction Δ for the rank is unlimited and possesses a mode, Δ increases for increasing value of the variate from zero up to unity. The correction is important for small numbers of observations. For large numbers of observations and for the ogive it is sufficient to choose $\Delta=1/2$. The calculation of Δ allows a correct plotting of all observations (including the first and last) on probability paper (equiprobability test). For the return periods, the ranks m and $m-1$ correspond to the observed exceedance and recurrence intervals. Generally the corrected return periods pass for increasing values of the variate from the exceedance to the recurrence intervals, provided the variate is unlimited and possesses a single mode. The asymptotic standard error of the partition values may be used to construct confidence bands for the ogive, the equiprobability test, and the return periods. This control for the fit between theory and observation may be applied to all observations which are not extreme. (Received July 30, 1943.)

239. Henry Scheffé: *On a measure-theoretic problem arising in the theory of non-parametric tests.*

Let μ be any measure on the real line, such that the measure of the whole line is unity, and form the "power" measure μ^k in Euclidean k-space—that is, the product

measure obtained by using μ on each axis. The problem is, for a given class C of measures μ, what can be said about the existence and structure of sets of points in the k-space which have the property that their "power" measure is the same for all measures μ in C? Necessary conditions and sufficient conditions for sets to have the above property are found for certain large classes C of statistical interest. (Received July 24, 1943.)

TOPOLOGY

240. H. D. Huskey: *On polyhedra and polyhedral path surfaces.*

In the study of surfaces in parametric form there are several definitions of polyhedra or of polyhedral path surfaces in common use. The equivalence of these definitions is discussed in this paper. A single polyhedral path surface may have several representations; the paper goes on to show directly that the area of a polyhedral path surface is independent of the representation (this result follows indirectly from results in the literature, see T. Radó: *On the semicontinuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361). Finally it is shown that the Lebesgue area of a surface is independent of the type of polyhedra used in the definition. (Received July 16, 1943.)

241. A. D. Wallace: *A structural property of transformations.*

Let M and N be compact connected Hausdorff spaces and let $f(M)=N$ be continuous. Suppose further that f satisfies the condition: If $M=A+B$ where A and B are closed and $A \cdot B$ is a point, then for any continuum K in M it is true that $f(K \cdot A) = f(K) \cdot f(A)$. The following results have been obtained: (1) If E is a prime-chain in N there exists a unique prime-chain F in M such that E is contained in $f(F)$ and (2) If X is a chain in M then $f(X)$ is a chain in N. For f non-alternating and metric M and N the above results are due respectively to G. T. Whyburn and G. E. Schweigert. If f is non-alternating then the above condition on f holds, but not conversely. Indeed f may be completely alternating. (Received July 28, 1943.)

242. Paul White: *On r-regular convergence.*

In this paper the notion of regular convergence as introduced by G. T. Whyburn (Fund. Math. vol. 25 (1935) is further studied. It is shown that a sequence of irreducible membranes of homologies or of irreducible carriers of r-dimensional cycles has as its limit under s-regular convergence ($s \leqq r$) a set of the same type. Finally it is shown that under this type of convergence a sequence of r-dimensional closed Cantorian manifolds converges to another r-dimensional closed Cantorian manifold. (Received July 2, 1943.)

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